Exercise 5B

1 a $(1\times2)\cdot(2\times2)=1\times2$

b $(3\times2)\cdot(2\times3)=3\times3$

c $(1\times3)\cdot(3\times2)=1\times2$

d $(2\times3)\cdot(3\times2)=2\times2$

e $(2\times2)\cdot(2\times3)=2\times3$

f $(3\times2)\cdot(2\times2)=3\times2$

2 a $\begin{pmatrix}1&2\\2&4\end{pmatrix}$ is a 2×2 matrix and

$\begin{pmatrix}-1\\2\end{pmatrix}$ is a 2×1 matix.

They can be multiplied and the product will be a 2×1 matrix.

$$\begin{pmatrix}1&2\\2&4\end{pmatrix}\times\begin{pmatrix}-1\\2\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$$

$$a=1\times(-1)+2\times2$$
$$=3$$
$$b=2\times(-1)+4\times2$$
$$=6$$

b $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}0&5\\-1&-2\end{pmatrix}=\begin{pmatrix}-2&1\\-4&7\end{pmatrix}$

3 a $\begin{pmatrix}-1&-2\\0&3\end{pmatrix}\begin{pmatrix}1&0&1\\1&1&0\end{pmatrix}=\begin{pmatrix}-3&-2&-1\\3&3&0\end{pmatrix}$

b $\begin{pmatrix}-1&-2\\0&3\end{pmatrix}\begin{pmatrix}-1&-2\\0&3\end{pmatrix}=\begin{pmatrix}1&-4\\0&9\end{pmatrix}$

4 a AB is $(2\times1)\cdot(2\times2)$ Not possible

b $\text{AC}=\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}-3&-2\end{pmatrix}=\begin{pmatrix}-6&-4\\-3&-2\end{pmatrix}$

c BC is $(2\times2)\cdot(1\times2)$ Not possible

d $\text{BA}=\begin{pmatrix}3&1\\-1&2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}7\\0\end{pmatrix}$

4 e $\text{CA}=\begin{pmatrix}-3&-2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}-8\end{pmatrix}$

f $\text{CB}=\begin{pmatrix}-3&-2\end{pmatrix}\begin{pmatrix}3&1\\-1&2\end{pmatrix}=\begin{pmatrix}-7&-7\end{pmatrix}$

5 $\begin{pmatrix}2&a\\1&-1\end{pmatrix}\begin{pmatrix}1&3&0\\0&-1&2\end{pmatrix}=\begin{pmatrix}2&6-a&2a\\1&4&-2\end{pmatrix}$

6 $\begin{pmatrix}3&2\\-1&x\end{pmatrix}\begin{pmatrix}x&-2\\1&3\end{pmatrix}=\begin{pmatrix}3x+2&0\\0&3x+2\end{pmatrix}$

7 a $\mathbf{AB}-\mathbf{C}=\begin{pmatrix}2&-1\\3&4\end{pmatrix}\begin{pmatrix}1&0\\-3&2\end{pmatrix}-\begin{pmatrix}-3&1\\1&2\end{pmatrix}$

$$=\begin{pmatrix}2\times1+(-1)\times(-3)&2\times0+(-1)\times2\\3\times1+4\times(-3)&3\times0+4\times2\end{pmatrix}-\begin{pmatrix}-3&1\\1&2\end{pmatrix}$$
$$=\begin{pmatrix}5&-2\\-9&8\end{pmatrix}-\begin{pmatrix}-3&1\\1&2\end{pmatrix}$$
$$=\begin{pmatrix}5+3&-2-1\\-9-1&8-2\end{pmatrix}$$
$$=\begin{pmatrix}8&-3\\-10&6\end{pmatrix}$$

b $\mathbf{BC}+3\mathbf{A}=\begin{pmatrix}1&0\\-3&2\end{pmatrix}\times\begin{pmatrix}-3&1\\1&2\end{pmatrix}+3\begin{pmatrix}2&-1\\3&4\end{pmatrix}$

$$=\begin{pmatrix}-3&1\\11&1\end{pmatrix}+\begin{pmatrix}6&-3\\9&12\end{pmatrix}$$
$$=\begin{pmatrix}3&-2\\20&13\end{pmatrix}$$

7 c $4\mathbf{B}-3\mathbf{CA}=4\begin{pmatrix}1&0\\-3&2\end{pmatrix}-3\begin{pmatrix}-3&1\\1&2\end{pmatrix}\begin{pmatrix}2&-1\\3&4\end{pmatrix}$

$$=\begin{pmatrix}4&0\\-12&8\end{pmatrix}-3\begin{pmatrix}-3&7\\8&7\end{pmatrix}$$
$$=\begin{pmatrix}4&0\\-12&8\end{pmatrix}-\begin{pmatrix}-9&21\\24&21\end{pmatrix}$$
$$=\begin{pmatrix}13&-21\\-36&-13\end{pmatrix}$$

8 a $\mathbf{MN} = \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} \times \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 3\times 1 + k\times k & 3\times k + k\times(-1) \\ k\times 1 + 1\times k & k\times k + 1\times(-1) \end{pmatrix}$

$= \begin{pmatrix} 3+k^2 & 2k \\ 2k & k^2-1 \end{pmatrix}$

b $\mathbf{NM} = \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix} \times \begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix}$

$= \begin{pmatrix} 1\times 3 + k\times k & 1\times k + k\times 1 \\ k\times 3 + (-1)\times k & k\times k + (-1)\times 1 \end{pmatrix}$

$= \begin{pmatrix} 3+k^2 & 2k \\ 2k & k^2-1 \end{pmatrix}$

c $3\mathbf{M} - 2\mathbf{N} = 3\begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 3k \\ 3k & 3 \end{pmatrix} - \begin{pmatrix} 2 & 2k \\ 2k & -2 \end{pmatrix}$

$= \begin{pmatrix} 7 & k \\ k & 5 \end{pmatrix}$

d $2\mathbf{NM} + 3\mathbf{N} = 2\begin{pmatrix} 3 & k \\ k & 1 \end{pmatrix} \times \begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix} + 3\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= 2\begin{pmatrix} 3+k^2 & 2k \\ 2k & k^2-1 \end{pmatrix} + 3\begin{pmatrix} 1 & k \\ k & -1 \end{pmatrix}$

$= \begin{pmatrix} 6+2k^2 & 4k \\ 4k & 2k^2-2 \end{pmatrix} + \begin{pmatrix} 3 & 3k \\ 3k & -3 \end{pmatrix}$

$= \begin{pmatrix} 9+2k^2 & 7k \\ 7k & 2k^2-5 \end{pmatrix}$

9 a $\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

b $\mathbf{A}^3 = \mathbf{AA}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

Note $\mathbf{A}^2 = \begin{pmatrix} 1 & 2\times 2 \\ 0 & 1 \end{pmatrix}$

$\mathbf{A}^3 = \begin{pmatrix} 1 & 2\times 3 \\ 0 & 1 \end{pmatrix}$

9 c Suggests $\mathbf{A}^k = \begin{pmatrix} 1 & 2\times k \\ 0 & 1 \end{pmatrix}$

10 a $\mathbf{A}^2 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix}$

b $\mathbf{A}^2 = 3\mathbf{A} \Rightarrow \begin{pmatrix} a^2 & 0 \\ ab & 0 \end{pmatrix} = \begin{pmatrix} 3a & 0 \\ 3b & 0 \end{pmatrix}$

$\Rightarrow \quad a^2 = 3a \quad \Rightarrow \quad a = 3 \quad (\text{or } 0)$

and $ab = 3b \quad \Rightarrow a = 3$

$\therefore \quad a = 3$

11 a $\mathbf{BAC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} -4 & -7 \end{pmatrix}$

$= \begin{pmatrix} -8 & -14 \\ -4 & -7 \\ 0 & 0 \end{pmatrix}$

b $\mathbf{AC}^2 = \begin{pmatrix} -1 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} -4 & -7 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 0 & -3 \end{pmatrix}$

$= \begin{pmatrix} -16 & 29 \end{pmatrix}$

12 a $\mathbf{ABA} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} -1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

12 b $\mathbf{BAB} = \begin{pmatrix} 3 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & -2 & -3 \end{pmatrix}$

$= (-1)\begin{pmatrix} 3 & -2 & -3 \end{pmatrix}$

$= \begin{pmatrix} -3 & 2 & 3 \end{pmatrix}$

13 a $\mathbf{I}_2$ is the 2×2 identity matrix in which the elements on the leading diagonal are all 1 and the remaining elements are 0.

$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b $\mathbf{AI} = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix} = \mathbf{A}$

$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$

$= \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix} = \mathbf{A}$

Hence, $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

14 $\mathbf{AB} + \mathbf{AC} = \mathbf{A}(\mathbf{B} + \mathbf{C})$

Consider LHS:

$\mathbf{AB} + \mathbf{AC}$

$= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 4 \\ 10 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} 11 & 9 \\ 13 & 10 \end{pmatrix}$

Consider RHS:

$\mathbf{A}(\mathbf{B} + \mathbf{C})$

$= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \times \left[\begin{pmatrix} 4 & 2 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \right]$

$= \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 5 & 4 \\ -1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 11 & 9 \\ 13 & 10 \end{pmatrix} = \mathbf{AB} + \mathbf{AC}$

15 $\mathbf{A}^2 = 2\mathbf{A} + 5\mathbf{I}$

Consider LHS:

$\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix}$

Consider RHS:

$2\mathbf{A} + 5\mathbf{I} = 2\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 4 \\ 6 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$= \begin{pmatrix} 7 & 4 \\ 6 & 7 \end{pmatrix} = \mathbf{A}^2$

16

$\mathbf{AB} = \mathbf{BA}$

Consider LHS:

$$\mathbf{AB} = \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix} \times \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix}$$

$$= \begin{pmatrix} pq+12 & 2p+3q \\ 4p+6q & 12+pq \end{pmatrix}$$

Consider RHS:

$$\mathbf{BA} = \begin{pmatrix} q & 2 \\ 4 & q \end{pmatrix} \times \begin{pmatrix} p & 3 \\ 6 & p \end{pmatrix}$$

$$= \begin{pmatrix} pq+12 & 2p+3q \\ 4p+6q & 12+pq \end{pmatrix} = \mathbf{AB}$$

17

$$\mathbf{A}^2 = \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix} \times \begin{pmatrix} 3 & p \\ -4 & q \end{pmatrix}$$

$$= \begin{pmatrix} 9-4p & 3p+pq \\ -12-4q & q^2-4p \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{so } \begin{pmatrix} 9-4p & 3p+pq \\ -12-4q & q^2-4p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Compare corresponding elements

$9-4p=1$

$4p=8 \Rightarrow p=2$

$3p+pq=0$

$6+2q=0$

$2q=-6 \Rightarrow q=-3$

Challenge

a

$$\mathbf{A} = \begin{pmatrix} (-1^k)mn & (-1^k)n^2 \\ -(-1^k)m^2 & -(-1^k)mn \end{pmatrix}$$

where n, m are real numbers and $k=0$ or 1.

$$\text{E.g. } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Challenge

b

$$\text{Let } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}$$

For $\mathbf{A}^2 = \mathbf{0}$ equate each element to 0

$a^2+bc=0$

$b(a+d)=0$

$c(a+d)=0$

$bc+d^2=0$

Set $a+d=0$ then $b=c=0$, subsequently giving $a=d=0$ (inadmissible).

So $a=-d$ and $bc=-a^2$ which has many solutions.

$$\text{E.g. } \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$