Chapter review 4

1 a A parabola of the form $y^2 = 4ax$ has the focus at $(a, 0)$
so $y^2 = 12x$ has the focus at $(3, 0)$

b $y = 3x$ intersects $y^2 = 12x$ at P where $y > 0$
Substituting $y = 3x$ into $y^2 = 12x$ gives:
$9x^2 = 12x$
$3x(3x - 4) = 0$
$x = 0$ or $x = \frac{4}{3}$

When $x = 0$, $y = 0$ and when $x = \frac{4}{3}$, $y = 4$

Since $y > 0$, P has coordinates $\left(\frac{4}{3}, 4\right)$

c $\text{Area}_{OPS} = \frac{1}{2}(3)(4)$
$= 6$

2 a $P(k, 6)$ lies on $y^2 = 24x$
Substituting $(k, 6)$ into $y^2 = 24x$ gives:
$(6)^2 = 24(k)$
$k = \frac{3}{2}$

b A parabola of the form $y^2 = 4ax$ has the focus at $(a, 0)$
so $y^2 = 24x$ has the focus at $(6, 0)$

c P has coordinates $\left(\frac{3}{2}, 6\right)$ and S has coordinates $(6, 0)$

PS has gradient:

$$m_{PS} = \frac{y_P - y_S}{x_P - x_S}$$
$$= \frac{6-0}{\frac{3}{2}-6}$$
$$= -\frac{4}{3}$$

Finding the equation of the tangent using
$y - y_1 = m(x - x_1)$ with $m = -\frac{4}{3}$ at $(6, 0)$ gives:

$y - 0 = -\frac{4}{3}(x - 6)$
$3y = -4x + 24$
$4x + 3y - 24 = 0$ as required

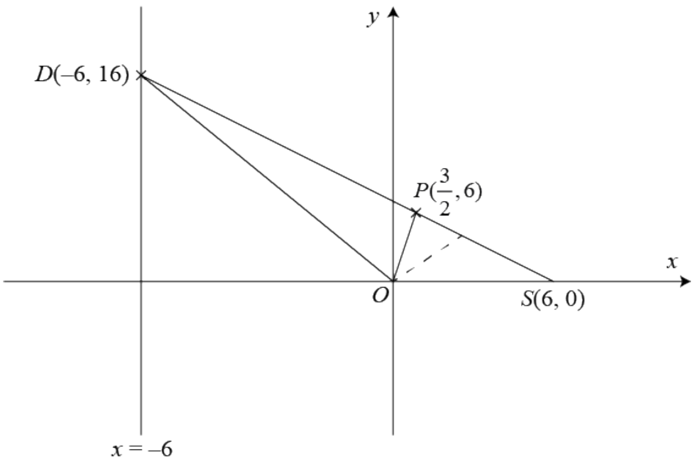
2 d A parabola of the form $y^2 = 4ax$ has the directrix at $x + a = 0$
so $y^2 = 24x$ has the directrix at $x + 6 = 0$
At the point D where the line l crosses the directrix $x = -6$
Substituting $x = -6$ into $4x+3y-24=0$ gives:

$4(-6)+3y-24=0$

$y=16$

So D has coordinates $(-6, 16)$

$P(\frac{3}{2},6)$



Let the point $(-6, 0)$ be C

$$\text{Area}_{ODP} = \text{Area}_{CDS} - \text{Area}_{CDO} - \text{Area}_{OPS}$$
$$= \frac{1}{2}(12)(16) - \frac{1}{2}(6)(16) - \frac{1}{2}(6)(6)$$
$$= 96 - 48 - 18$$
$$= 30$$

3 a $x = 12t^2,\ y = 24t$

The general form of a parabola is $y^2 = 4ax$

Substituting $x = 12t^2$ and $y = 24t$ into $y^2 = 4ax$ gives:

$$(24t)^2 = 4a(12t^2)$$
$$576 = 48a$$
$$a = 12$$

So the parabola is $y^2 = 48x$

b A parabola of the form $y^2 = 4ax$ has the directrix at $x + a = 0$
so $y^2 = 48x$ has the directrix at $x + 12 = 0$

3 c The general form of a point on the parabola is $\left(x, \pm 4\sqrt{3x}\right)$

The distance from any point on a parabola to the focus is the same as the distance from the point to the directrix. PS has length 28 so the distance from P to the directrix is also 28.

$x + 12 = 28$

$x = 16$

We are told that $y > 0$, so P is the point $\left(16, 4\sqrt{3(16)}\right) = \left(16, 16\sqrt{3}\right)$

d $\text{Area}_{OSP} = \frac{1}{2}(12)\left(16\sqrt{3}\right)$

$= 96\sqrt{3}$

4 a The line l with equation $4x - 9y + 32 = 0$ meets $y^2 = 16x$ at P and Q

$y^2 = 16x \Rightarrow x = \frac{y^2}{16}$

Substituting $x = \frac{y^2}{16}$ into $4x - 9y + 32 = 0$ gives:

$4\left(\frac{y^2}{16}\right) - 9y + 32 = 0$

$y^2 - 36y + 128 = 0$

$(y - 4)(y - 32) = 0$

$y = 4$ or $y = 32$

When $y = 4$, $x = 1$ and when $y = 32$, $x = 64$

So P and Q have coordinates (1, 4) and (64, 32)

b $y^2 = 16x \Rightarrow y = 4x^{\frac{1}{2}}$

The gradient of the tangent is

$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$

$= \frac{2}{x^{\frac{1}{2}}}$

At $x = 4t^2$

$\frac{dy}{dx} = \frac{2}{\left(4t^2\right)^{\frac{1}{2}}}$

$= \frac{1}{t}$

At $x = 4t^2$ the tangent has gradient $\frac{1}{t}$ so the normal has gradient $-t$

Finding the equation of the normal using

$y - y_1 = m(x - x_1)$ with $m = -t$ at $\left(4t^2, 8t\right)$ gives:

$y - 8t = -t\left(x - 4t^2\right)$

$y - 8t = -xt + 4t^3$

$xt + y = 4t^3 + 8t$ as required

4 c P and Q have coordinates (1, 4) and (64, 32)
Comparing the y-coordinates of (1, 4) with $\left(4t^2, 8t\right)$ gives:

$8t = 4 \Rightarrow t = \frac{1}{2}$

Substituting $t = \frac{1}{2}$ into $xt + y = 4t^3 + 8t$ gives:

$\left(\frac{1}{2}\right)x + y = 4\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)$

$\frac{1}{2}x + y = \frac{1}{2} + 4$

$x + 2y - 9 = 0$

Comparing the y-coordinates of (64, 32) with $\left(4t^2, 8t\right)$ gives:

$8t = 32 \Rightarrow t = 4$

Substituting $t = 4$ into $xt + y = 4t^3 + 8t$ gives:

$(4)x + y = 4(4)^3 + 8(4)$

$4x + y - 288 = 0$

So the equations of the normals at P and Q are
$x + 2y - 9 = 0$ and $4x + y - 288 = 0$

d The normals at P and Q meet at R
$x + 2y - 9 = 0 \Rightarrow x = -2y + 9$

$4x + y - 288 = 0 \Rightarrow x = -\frac{1}{4}y + 72$

Equating these equations gives:

$-2y + 9 = -\frac{1}{4}y + 72$

$8y - 36 = y - 288$

$7y = -252$

$y = -36$

When $y = -36$, $x = 81$ so R has coordinates $(81, -36)$

e $|OR| = \sqrt{(81-0)^2 + (-36-0)^2}$

$= 9\sqrt{97}$

5 a $P\left(at^2, 2at\right)$ lies on $y^2 = 4ax$
A parabola of the form $y^2 = 4ax$ has the focus at $(a, 0)$
A parabola of the form $y^2 = 4ax$ has the directrix at $x + a = 0$
So Q has coordinates $(-a, 0)$

5 b $y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of the tangent is

$$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$$

$$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$$

At $x = at^2$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{\left(at^2\right)^{\frac{1}{2}}}$$

$$= \frac{1}{t}$$

Finding the equation of the tangent using

$y - y_1 = m\left(x - x_1\right)$ with $m = \frac{1}{t}$ at $\left(at^2, 2at\right)$ gives:

$$y - 2at = \frac{1}{t}\left(x - at^2\right)$$

$$ty - 2at^2 = x - at^2$$

$$x - ty + at^2 = 0$$

Substituting $\left(-a, 0\right)$ into $x - ty + at^2 = 0$ gives:

$$-a - t\left(0\right) + at^2 = 0$$

$$at^2 = a$$

$$t = \pm 1$$

Substituting $t = 1$ into $\left(at^2, 2at\right)$ gives $(a, 2a)$

Substituting $t = -1$ into $\left(at^2, 2at\right)$ gives $(a, -2a)$

So P has coordinates $(a, 2a)$ or $(a, -2a)$

6 a $P\left(ct, \frac{c}{t}\right)$, $c > 0$, $t \neq 0$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2x^{-1}$

The gradient of a tangent to a point on the hyperbola is:

$$\frac{dy}{dx} = -c^2x^{-2}$$
$$= -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{(ct)^2}$$
$$= -\frac{1}{t^2}$$

At $x = ct$ the gradient of the tangent is $-\frac{1}{t^2}$ so the gradient of the normal is t^2

Finding the equation of the normal using

$y - y_1 = m(x - x_1)$ with $m = t^2$ at $\left(ct, \frac{c}{t}\right)$ gives:

$$y - \frac{c}{t} = t^2(x - ct)$$
$$ty - c = t^3x - ct^4$$
$t^3x - ty = c(t^4 - 1)$ as required

b (12, 3) lies on $xy = 36$

$P\left(ct, \frac{c}{t}\right)$ lies on $xy = c^2$

Comparing $\left(ct, \frac{c}{t}\right)$ to (12, 3) gives:

$ct = 12$ and $\frac{c}{t} = 3$

Solving gives $c = 6$ and $t = 2$

Substituting these values into the equation of the normal found in part **a** gives:

$$t^3x - ty = c(t^4 - 1)$$
$$(2)^3x - (2)y = (6)((2)^4 - 1)$$
$$4x - y - 45 = 0$$

6 c $4x - y - 45 = 0$ meets the curve $xy = 36$ again at Q

Substituting $y = \frac{36}{x}$ into $4x - y - 45 = 0$ gives:

$4x - \left(\frac{36}{x}\right) - 45 = 0$

$4x^2 - 45x - 36 = 0$

$(4x + 3)(x - 12) = 0$

$x = -\frac{3}{4}$ or $x = 12$

When $x = -\frac{3}{4}$, $y = -48$ so Q has coordinates $\left(-\frac{3}{4}, -48\right)$

7 The hyperbola $xy = 9$ has tangents l_1 and l_2 both with gradient $-\frac{1}{4}$

$xy = 9 \Rightarrow y = 9x^{-1}$

The gradient of a tangent to a point on the hyperbola is:

$\frac{dy}{dx} = -9x^{-2}$

$= -\frac{9}{x^2}$

l_1 and l_2 have gradient $-\frac{1}{4}$ so

$-\frac{9}{x^2} = -\frac{1}{4}$

$x = \pm 6$

When $x = 6$, $y = \frac{3}{2}$ and when $x = -6$, $y = -\frac{3}{2}$

So the points where l_1 and l_2 touch the hyperbola are $\left(6, \frac{3}{2}\right)$ and $\left(-6, -\frac{3}{2}\right)$

Finding the equations of l_1 and l_2 using

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{4}$ at $\left(6, \frac{3}{2}\right)$ gives:

$y - \frac{3}{2} = -\frac{1}{4}(x - 6)$

$4y - 6 = -x + 6$

$x + 4y - 12 = 0$

Using $y - y_1 = m(x - x_1)$ with $m = -\frac{1}{4}$ at $\left(-6, -\frac{3}{2}\right)$ gives:

$y + \frac{3}{2} = -\frac{1}{4}(x + 6)$

$4y + 6 = -x - 6$

$x + 4y + 12 = 0$

So l_1 and l_2 have the equations $x + 4y - 12 = 0$ and $x + 4y + 12 = 0$

8 **a** $P\left(ct, \frac{c}{t}\right)$, $c > 0$, $t \neq 0$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2x^{-1}$

The gradient of a tangent to a point on the hyperbola is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2}$$

$$= -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct)^2}$$

$$= -\frac{1}{t^2}$$

Finding the equation of the tangent using

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{t^2}$ at $\left(ct, \frac{c}{t}\right)$ gives:

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = -x + ct$$

$$x + t^2y - 2ct = 0$$

At X, $y = 0$

Substituting $y = 0$ into $x + t^2y - 2ct = 0$ gives:

$$x + t^2(0) - 2ct = 0$$

$x = 2ct$ so X has coordinates $(2ct, 0)$

At Y, $x = 0$

Substituting $x = 0$ into $x + t^2y - 2ct = 0$ gives:

$$(0) + t^2y - 2ct = 0$$

$y = \frac{2c}{t}$ so Y has coordinates $\left(0, \frac{2c}{t}\right)$

b O is the point (0, 0) and the triangle OXY has an area of 144

$$\text{Area}_{OXY} = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = 144$$

$2c^2 = 144$

$c = \pm 6\sqrt{2}$

Since $c > 0$

$c = 6\sqrt{2}$

9 a $P\left(ct, \frac{c}{t}\right)$, $c > 0$, $t \neq 0$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2 x^{-1}$

The gradient of a tangent to a point on the hyperbola is:

$$\frac{dy}{dx} = -c^2 x^{-2}$$

$$= -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{(ct)^2}$$

$$= -\frac{1}{t^2}$$

Finding the equation of the tangent using

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{t^2}$ at $\left(ct, \frac{c}{t}\right)$ gives:

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$x + t^2 y = 2ct$ as required

b $x + t^2 y = 2ct$ cuts the x-axis at $(2a, 0)$

Substituting $(2a, 0)$ into $x + t^2 y = 2ct$ gives:

$$(2a) + t^2(0) = 2ct$$

$$t = \frac{a}{c}$$

Substituting $t = \frac{a}{c}$ into $\left(ct, \frac{c}{t}\right)$ gives $\left(a, \frac{c^2}{a}\right)$ as required

c The x-coordinate of Q is $2a$

Substituting $x = 2a$ into $xy = c^2$ gives:

$$y = \frac{c^2}{2a}$$

9 d OQ has gradient

$$m_{OQ} = \frac{y_Q - y_O}{x_Q - x_O}$$
$$= \frac{\frac{c^2}{2a} - 0}{2a - 0}$$
$$= \frac{c^2}{4a^2}$$

Finding the equation of the OQ using

$y - y_1 = m(x - x_1)$ with $m = \dfrac{c^2}{4a^2}$ at (0, 0) gives:

$$y - 0 = \frac{c^2}{4a^2}(x - 0)$$
$$y = \frac{c^2 x}{4a^2}$$

e From **a**, the equation of the tangent to H at $\left(ct, \dfrac{c}{t}\right)$ is $x + t^2 y = 2ct$. PX is the tangent at P.

From **b**, at P, $t = \dfrac{a}{c}$

Therefore, the equation of PX is $x + \left(\dfrac{a}{c}\right)^2 y = 2c\left(\dfrac{a}{c}\right)$

$\Rightarrow y = \dfrac{2ac^2 - c^2 x}{a^2}$ or $y = -\dfrac{c^2}{a^2}x + \dfrac{2c^2}{a}$

From **d**, OQ has the equation $y = \dfrac{c^2 x}{4a^2}$

Equating gives:

$$\frac{2ac^2 - c^2 x}{a^2} = \frac{c^2 x}{4a^2}$$
$$4(2ac^2 - c^2 x) = c^2 x$$
$$8ac^2 - 4c^2 x = c^2 x$$
$$5c^2 x = 8ac^2$$
$$x = \frac{8a}{5}$$

When $x = \dfrac{8a}{5}$, $y = \dfrac{2c^2}{5a}$ so R has coordinates $\left(\dfrac{8a}{5}, \dfrac{2c^2}{5a}\right)$

9 f OQ and XP are perpendicular, therefore $m_{OQ} \times m_{XP} = -1$

From part **d** $m_{OQ} = \dfrac{c^2}{4a^2}$

From part **e** $m_{XP} = -\dfrac{c^2}{a^2}$

$$\frac{c^2}{4a^2} \times -\frac{c^2}{a^2} = -1$$

$$\frac{c^4}{4a^4} = 1$$

$$c^4 = 4a^4$$

$c^2 = 2a^2$ as required

g From part **e** R has y-coordinate $\dfrac{2c^2}{5a}$

Substituting $c^2 = 2a^2$ gives:

$$\frac{2c^2}{5a} = \frac{2(2a^2)}{5a}$$

$$= \frac{4a}{5}$$

Challenge

For the parabola with equation $x^2 = 4ay$

$x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

The gradient of a tangent to a point on the parabola is:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{2a}$

Let the point where the tangent meets $x^2 = 4ay$ be (x_1, y_1)

Since (x_1, y_1) lies on $x^2 = 4ay$ it is of the form $\left(x_1, \frac{x_1^{\ 2}}{4a}\right)$

Therefore the gradient at x_1 is $\frac{x_1}{2a}$

The equation of the tangent to $x^2 = 4ay$ is found using:

$y - y_1 = m(x - x_1)$ with $m = \frac{x_1}{2a}$ at $\left(x_1, \frac{x_1^{\ 2}}{4a}\right)$

$y - \frac{x_1^{\ 2}}{4a} = \frac{x_1}{2a}(x - x_1)$

$y = \frac{x_1 x}{2a} - \frac{x_1^{\ 2}}{2a} + \frac{x_1^{\ 2}}{4a}$

$y = \frac{x_1 x}{2a} - \frac{x_1^{\ 2}}{4a}$

For the parabola with equation $y^2 = 4ax$

$y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of a tangent to a point on the parabola is:

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{\sqrt{x}}$

Let the point where the tangent meets $y^2 = 4ax$ be (x_2, y_2)

Since (x_2, y_2) lies on $y^2 = 4ax$ it is of the form $\left(x_2, 2\sqrt{ax_2}\right)$

Therefore the gradient at x_2 is $\frac{\sqrt{a}}{\sqrt{x_2}}$

The equation of the tangent to $y^2 = 4ax$ is found using:

$y - y_2 = m(x - x_2)$ with $m = \frac{\sqrt{a}}{\sqrt{x_2}}$ at $\left(x_2, 2\sqrt{ax_2}\right)$

$y - 2\sqrt{ax_2} = \frac{\sqrt{a}}{\sqrt{x_2}}(x - x_2)$

$y = \frac{\sqrt{ax}}{\sqrt{x_2}} - \sqrt{ax_2} + 2\sqrt{ax_2}$

$y = \frac{\sqrt{ax}}{\sqrt{x_2}} + \sqrt{ax_2}$

So the equations of the tangent are:

$$y = \frac{x_1 x}{2a} - \frac{x_1^{\ 2}}{4a} \text{ and } y = \frac{\sqrt{a}x}{\sqrt{x_2}} + \sqrt{ax_2}$$

Since these represent the same line, comparing coefficients gives:

$$-\frac{x_1}{2a} = \frac{\sqrt{a}}{\sqrt{x_2}} \text{ and } \frac{x_1^{\ 2}}{4a} = \sqrt{ax_2}$$

$$\frac{x_1^{\ 2}}{4a} = \sqrt{ax_2} \Rightarrow \sqrt{x_2} = \frac{x_1^{\ 2}}{4a\sqrt{a}}$$

Substituting $\sqrt{x_2} = \frac{x_1^{\ 2}}{4a\sqrt{a}}$ into $-\frac{x_1}{2a} = \frac{\sqrt{a}}{\sqrt{x_2}}$ gives:

$$-\frac{x_1}{2a} = \frac{\sqrt{a}}{\left(\frac{x_1^{\ 2}}{4a\sqrt{a}}\right)}$$

$$-\frac{x_1}{2a} = \frac{4a^2}{x_1^{\ 2}}$$

$$x_1^{\ 3} = -8a^3$$

$$x_1 = -2a$$

Since (x_1, y_1) is of the form $\left(x_1, \frac{x_1^{\ 2}}{4a}\right)$

Substituting $x_1 = -2a$ into $y = \frac{x_1 x}{2a} - \frac{x_1^{\ 2}}{4a}$ (the equation of the tangent to $x^2 = 4ay$) gives:

$$y = \frac{(-2a)x}{2a} - \frac{(-2a)^2}{4a}$$

$$= -x - a$$

So the equation of the line that is tangent to both parabolas is $y + x + a = 0$