Exercise 4E

1 a $P\left(3t^2, 6t\right)$ lies on $y^2 = 12x$

$y = 2\sqrt{3}x^{\frac{1}{2}}$

The gradient is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{3}\left(x\right)^{-\frac{1}{2}}$

When $x = 3t^2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{3}}{\left(3t^2\right)^{\frac{1}{2}}}$

$= \frac{\sqrt{3}}{\sqrt{3}t}$

$= \frac{1}{t}$

Find the equation of the tangent using:

$y - y_1 = m\left(x - x_1\right)$ with $m = \frac{1}{t}$ at $\left(3t^2, 6t\right)$

Gives:

$y - 6t = \frac{1}{t}\left(x - 3t^2\right)$

$yt - 6t^2 = x - 3t^2$

$yt = x + 3t^2$ as required

b At $P\left(3t^2, 6t\right)$, the gradient of the tangent is $\frac{1}{t}$ so the gradient of the normal is $-t$

Finding the equation of the normal using

$y - y_1 = m\left(x - x_1\right)$ with $m = -t$ at $\left(3t^2, 6t\right)$

gives:

$y - 6t = -t\left(x - 3t^2\right)$

$xt + y = 3t^3 + 6t$ as required

2 a $P\left(6t, \frac{6}{t}\right)$, $t \neq 0$ lies on $xy = 36$

$y = 36x^{-1}$

The gradient is

$\frac{\mathrm{d}y}{\mathrm{d}x} = -36x^{-2}$

When $x = 6t$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -36\left(6t\right)^{-2}$

$= -\frac{36}{\left(6t\right)^2}$

$= -\frac{1}{t^2}$

Find the equation of the tangent using:

$y - y_1 = m\left(x - x_1\right)$ with $m = -\frac{1}{t^2}$ at $\left(6t, \frac{6}{t}\right)$

Gives:

$y - \frac{6}{t} = -\frac{1}{t^2}\left(x - 6t\right)$

$t^2y - 6t = -x + 6t$

$x + t^2y = 12t$ as required

b At $P\left(6t, \frac{6}{t}\right)$, the gradient of the tangent is $-\frac{1}{t^2}$ so the gradient of the normal is t^2

Finding the equation of the normal using

$y - y_1 = m\left(x - x_1\right)$ with $m = t^2$ at $\left(6t, \frac{6}{t}\right)$

gives:

$y - \frac{6}{t} = t^2\left(x - 6t\right)$

$ty - 6 = t^3x - 6t^4$

$t^3x - ty = 6\left(t^4 - 1\right)$ as required

3 a $P(5t^2, 10t)$, $t \neq 0$ lies on $y^2 = 4ax$

Substituting $(5t^2, 10t)$ into $y^2 = 4ax$ gives:

$(10t)^2 = 4a(5t^2)$

$100t^2 = 20at^2$

$a = 5$

b The parabola has equation

$y^2 = 20x \Rightarrow 2\sqrt{5}x^{\frac{1}{2}}$

The gradient of a tangent to the parabola is:

$\frac{dy}{dx} = \sqrt{5}x^{-\frac{1}{2}}$

At $x = 5t^2$

$\frac{dy}{dx} = \frac{\sqrt{5}}{(5t^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{t}$ at $(5t^2, 10t)$ gives:

$y - 10t = \frac{1}{t}(x - 5t^2)$

$ty - 10t^2 = x - 5t^2$

$yt = x + 5t^2$ as required

c The tangent cuts the x-axis at $y = 0$ therefore

$t(0) = x + 5t^2$

$x = -5t^2$

So X has coordinates $(-5t^2, 0)$

The tangent cuts the y-axis at $x = 0$ therefore

$ty = 0 + 5t^2$

$y = 5t$

So Y has coordinates $(0, 5t)$

$\text{Area}_{OXY} = \left|\frac{1}{2}(-5t^2)(5t)\right|$

$= |12.5t^3|$

4 a $P(at^2, 2at)$, $t \neq 0$ lies on $y^2 = 4ax$

$y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient is

$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$

$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$

When $x = at^2$

$\frac{dy}{dx} = \frac{\sqrt{a}}{(at^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{t}$ at $(at^2, 2at)$ gives:

$y - 2at = \frac{1}{t}(x - at^2)$

$ty - 2at^2 = x - at^2$

$ty = x + at^2$ as required

b From **a**, the tangent to any point, $(at^2, 2at)$, on the curve has equation $ty = x + at^2$.

The point $(-4a, 3a)$ lies on the tangents at A and B.

Thus, for A and B:

$t(3a) = (-4a) + at^2$

$at^2 - 3at - 4a = 0$

$t^2 - 3t - 4 = 0$

$(t + 1)(t - 4) = 0$

$t = -1$ or $t = 4$

Therefore, the co-ordinates of A and B are $(a, -2a)$ and $(16a, 8a)$.

5 a $P\left(4t, \frac{4}{t}\right)$, $t \neq 0$ lies on $xy = 16$

$xy = 16 \Rightarrow y = 16x^{-1}$

The gradient of a tangent to the hyperbola is

$$\frac{dy}{dx} = -16x^{-2}$$
$$= -\frac{16}{x^2}$$

At $x = 4t$

$$\frac{dy}{dx} = -\frac{16}{(4t)^2}$$
$$= -\frac{1}{t^2}$$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = -\frac{1}{t^2}$ at $\left(4t, \frac{4}{t}\right)$ gives:

$$y - \frac{4}{t} = -\frac{1}{t^2}(x - 4t)$$
$$t^2 y - 4t = -x + 4t$$
$$x + t^2 y = 8t \text{ as required}$$

b The directrix of a parabola of the form $y^2 = 4ax$ is at $x + a = 0$,

so the directrix of $y^2 = 16x$ is at $x + 4 = 0$

hence X has coordinates $(-4, 5)$

5 c From **a**, the tangent to any point, $(4t^2, 8t)$, on the curve has equation $x + t^2 y = 8t$.

$X = (-4, 5)$ lies on the tangents at A and B.

Thus, for A and B:

$$-4 + 5t^2 = 8t$$
$$5t^2 - 8t - 4 = 0$$
$$(5t + 2)(t - 2) = 0$$
$$t = -\frac{2}{5} \text{ or } t = 2$$

Therefore, the co-ordinates of A and B are $(-\frac{8}{5}, -10)$ and $(8, 2)$.

d From **c**, for the tangents to H which pass through X, $t = -\frac{2}{5}$ or $t = 2$.

Using the equation found in **a**, $x + t^2 y = 8t$ gives:

$$x + \left(-\frac{2}{5}\right)^2 y = 8\left(-\frac{2}{5}\right) \text{ and}$$
$$x + (2)^2 y = 8(2)$$

So, the equations of the tangents are:

$$25x + 4y + 80 = 0 \text{ and } x + 4y - 16 = 0$$

6 a $P(at^2, 2at),\ t \neq 0$ lies on $y^2 = 4ax$

$y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of the tangent is

$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$

$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$

When $x = at^2$

$\frac{dy}{dx} = \frac{\sqrt{a}}{(at^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{t}$ at $(at^2, 2at)$ gives:

$y - 2at = \frac{1}{t}(x - at^2)$

$ty - 2at^2 = x - at^2$

$ty = x + at^2$

At the point where the tangent cuts the x-axis $y = 0$ so

$t(0) = x + at^2$

$x = -at^2$

Therefore A has coordinates $(-at^2, 0)$

b At $x = at^2$ the gradient of the tangent is $\frac{1}{t}$

so the gradient of the normal is $-t$

Finding the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = -t$ at $(at^2, 2at)$ gives:

$y - 2at = -t(x - at^2)$

$tx + y = at^3 + 2at$

At the point where the tangent cuts the x-axis $y = 0$ so

$tx + (0) = at^3 + 2at$

$x = at^2 + 2a$

Therefore B has coordinates $(at^2 + 2a, 0)$

6 c $\text{Area}_{APB} = \frac{1}{2}(at^2 + 2a - (-at^2))(2at)$

$= 2a^2t(t^2 + 1)$

7 a $P(2t^2, 4t),\ t \neq 0$ lies on $y^2 = 8x$

$y^2 = 8x \Rightarrow y = 2\sqrt{2}x^{\frac{1}{2}}$

The gradient of the tangent is

$\frac{dy}{dx} = \sqrt{2}x^{-\frac{1}{2}}$

$= \frac{\sqrt{2}}{x^{\frac{1}{2}}}$

When $x = 2t^2$

$\frac{dy}{dx} = \frac{\sqrt{2}}{(2t^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

At $x = 2t^2$ the gradient of the tangent is $\frac{1}{t}$

so the gradient of the normal is $-t$

Finding the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = -t$ at $(2t^2, 4t)$ gives:

$y - 4t = -t(x - 2t^2)$

$y - 4t = -tx + 2t^3$

$xt + y = 2t^3 + 4t$ as required

b Since (12, 0) lies on $xt + y = 2t^3 + 4t$

$(12)t + (0) = 2t^3 + 4t$

$2t^3 - 8t = 0$

$2t(t^2 - 4) = 0$

$t = 0$ or $t = \pm 2$

To find the coordinates of R, S and T, substitute $t = 0$, 2 and -2 into $(2t^2, 4t)$:

When $t = 0$

(0, 0)

When $t = 2$

(8, 8)

When $t = -2$

(8, −8)

7 c To find the equations of the normal at R, S and T, substitute $t = 0$, 2 and -2 into the general equation of the normal:

When $t = 0$

$x(0) + y = 2(0)^3 + 4(0)$

$y = 0$

When $t = 2$

$x(2) + y = 2(2)^3 + 4(2)$

$2x + y - 24 = 0$

When $t = -2$

$x(-2) + y = 2(-2)^3 + 4(-2)$

$2x - y - 24 = 0$

So the equations of the normal that pass through (12, 0) are:

$y = 0,\ 2x + y - 24 = 0$ and $2x - y - 24 = 0$

8 a $P(at^2, 2at),\ t \neq 0$ lies on $y^2 = 4ax$

$y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of the tangent is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{a}x^{-\frac{1}{2}}$

$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$

When $x = at^2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{a}}{(at^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

So at $x = at^2$ the gradient of the tangent is

$= \frac{1}{t}$

Finding the equation of the tangent using

$y - y_1 = m(x - x_1)$ with $m = \frac{1}{t}$ at $(at^2, 2at)$ gives:

$y - 2at = \frac{1}{t}(x - at^2)$

$x - yt + at^2 = 0$

The tangent cuts the y-axis at $x = 0$, so

$x(0) - yt + at^2 = 0$

$y = at$

So Q has coordinates $(0, at)$

8 b The focus of a parabola of the form $y^2 = 4ax$ is at $(a,\ 0)$

c P is the point $(at^2, 2at)$, Q is the point $(0, at)$ and S is the point $(a, 0)$

From **a**, the gradient of PS is $\frac{1}{t}$.

The gradient of SQ is found using:

$m_{SQ} = \frac{y_S - y_Q}{x_S - x_Q}$

$= \frac{0 - at}{a - 0}$

$= -t$

$m_{PQ} \times m_{SQ} = \frac{1}{t} \times -t$

$= -1$

Therefore PQ and SQ are perpendicular.

9 a $P(6t^2, 12t),\ t \neq 0$ lies on $y^2 = 24x$

$y^2 = 24x \Rightarrow y = 2\sqrt{6}x^{\frac{1}{2}}$

The gradient of the tangent is

$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{6}x^{-\frac{1}{2}}$

$= \frac{\sqrt{6}}{x^{\frac{1}{2}}}$

When $x = 6t^2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{6}}{(6t^2)^{\frac{1}{2}}}$

$= \frac{1}{t}$

So at $x = 6t^2$ the gradient of the tangent is $\frac{1}{t}$

Finding the equation of the tangent using

$y - y_1 = m(x - x_1)$ with $m = \frac{1}{t}$ at $(6t^2, 12t)$ gives:

$y - 12t = \frac{1}{t}(x - 6t^2)$

$ty = x + 6t^2$ as required

9 b The directrix of a parabola of the form $y^2 = 4ax$ is at $x + a = 0$

so the directrix of $y^2 = 24x$ is at $x + 6 = 0$

therefore X has coordinates $(-6, 9)$

c $(-6, 9)$ lies on the tangent through B with general equation $ty = x + 6t^2$

Thus, $9t = -6 + 6t^2$

$6t^2 - 9t - 6 = 0$

$2t^2 - 3t - 2 = 0$

$(2t + 1)(t - 2) = 0$

$t = -\frac{1}{2}$ or 2

To find the possible coordinates of B, substitute $t = -\frac{1}{2}$ and $t = 2$ into $\left(6t^2, 12t\right)$

So, $B = \left(\frac{3}{2}, -6\right)$ or $(24, 24)$