Exercise 4D

1 a $y^2 = 4x \Rightarrow y = 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

At (16, 8)

$\frac{dy}{dx} = \frac{1}{\sqrt{16}}$

$= \frac{1}{4}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{4}$ at (16, 8) gives:

$y - 8 = \frac{1}{4}(x - 16)$

$x - 4y + 16 = 0$

b $y^2 = 8x \Rightarrow y = 2\sqrt{2}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \sqrt{2}x^{-\frac{1}{2}}$

$= \frac{\sqrt{2}}{\sqrt{x}}$

At $(4, 4\sqrt{2})$

$\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{4}}$

$= \frac{\sqrt{2}}{2}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = \frac{\sqrt{2}}{2}$ at $(4, 4\sqrt{2})$ gives:

$y - 4\sqrt{2} = \frac{\sqrt{2}}{2}(x - 4)$

$\sqrt{2}x - 2y + 4\sqrt{2} = 0$

or $x - \sqrt{2}y + 4 = 0$

1 c $xy = 25 \Rightarrow y = \frac{25}{x} \Rightarrow y = 25x^{-1}$

$\frac{dy}{dx} = -25x^{-2}$

At (5, 5)

$\frac{dy}{dx} = -25(5)^{-2}$

$= -\frac{25}{25}$

$= -1$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = -1$ at (5, 5) gives:

$y - 5 = -(x - 5)$

$x + y - 10 = 0$

d $xy = 4 \Rightarrow y = \frac{4}{x} \Rightarrow y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2}$

At $x = \frac{1}{2}, y = 8$

$\frac{dy}{dx} = -4\left(\frac{1}{2}\right)^{-2}$

$= -16$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = -16$ at $\left(\frac{1}{2}, 8\right)$ gives:

$y - 8 = -16\left(x - \frac{1}{2}\right)$

$16x + y - 16 = 0$

1 e $y^2 = 7x \Rightarrow y = \pm\sqrt{7}x^{\frac{1}{2}}$

At (7, −7) we need to take the negative square root to give $y = -\sqrt{7}x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{\sqrt{7}}{2}x^{-\frac{1}{2}}$

At $(7,-7)$

$\frac{dy}{dx} = -\frac{\sqrt{7}}{2}(7)^{-\frac{1}{2}}$

$= -\frac{\sqrt{7}}{2\sqrt{7}}$

$= -\frac{1}{2}$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = -\frac{1}{2}$ at $(7,-7)$ gives:

$y + 7 = -\frac{1}{2}(x-7)$

$x + 2y + 7 = 0$

f $xy = 16 \Rightarrow y = \frac{16}{x} \Rightarrow y = 16x^{-1}$

$\frac{dy}{dx} = -16x^{-2}$

At $x = 2\sqrt{2}$, $y = 4\sqrt{2}$

$\frac{dy}{dx} = -16(2\sqrt{2})^{-2}$

$= \frac{-16}{(2\sqrt{2})^2}$

$= -2$

Finding the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = -2$ at $(2\sqrt{2}, 4\sqrt{2})$ gives:

$y - 4\sqrt{2} = -2(x - 2\sqrt{2})$

$2x + y - 8\sqrt{2} = 0$

2 a $y^2 = 20x \Rightarrow y = 2\sqrt{5}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \sqrt{5}x^{-\frac{1}{2}}$

When $y = 10$, $x = 5$ so

$\frac{dy}{dx} = \sqrt{5}(5)^{-\frac{1}{2}}$

$= 1$

At $x = 5$ the gradient of the tangent is 1 so the gradient of the normal is −1.

Finding the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = -1$ at (5, 10) gives:

$y - 10 = -1(x-5)$

$x + y - 15 = 0$

b $xy = 9 \Rightarrow y = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2}$

At $\left(-\frac{3}{2}, -6\right)$

$\frac{dy}{dx} = -9\left(-\frac{3}{2}\right)^{-2}$

$= -\frac{9}{\left(-\frac{3}{2}\right)^2}$

$= -4$

At $x = -\frac{3}{2}$ the gradient of the tangent is −4 so the gradient of the normal is $\frac{1}{4}$.

Finding the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{4}$ at $\left(-\frac{3}{2}, -6\right)$ gives:

$y + 6 = \frac{1}{4}\left(x + \frac{3}{2}\right)$

$2x - 8y - 45 = 0$

3 a Since the point $P(4, 8)$ lies on $y^2 = 4ax$

$8^2 = 4(4)a$

$a = 4$

3 b $y^2 = 16x \Rightarrow y = \pm 4x^{\frac{1}{2}}$

At (4, 8), $y = 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2x^{-\frac{1}{2}}$

When $x = 4$

$\frac{dy}{dx} = 2(4)^{-\frac{1}{2}}$

$= \frac{2}{2} = 1$

At $x = 4$ the gradient of the tangent is 1 so the gradient of the normal is -1.

Find the equation of the normal using

$y - y_1 = m(x - x_1)$ with $m = -1$ at (4, 8)

gives:

$y - 8 = -(x - 4)$

$x + y - 12 = 0$

c Q lies on the curve $y^2 = 16x$ and the line $x + y - 12 = 0$ or $y = 12 - x$

Therefore, at Q, $(12 - x)^2 = 16x$

$144 - 24x + x^2 = 16x$

$x^2 - 40x + 144 = 0$

$(x - 4)(x - 36) = 0$

$x = 4$ or $x = 36$

$x = 4$ gives the point P

So, at Q, $x = 36$ and $y = 12 - 36 = -24$

Thus $Q = (36, -24)$

d The length of PQ

$= \sqrt{(36-4)^2 + (-24-8)^2}$

$= \sqrt{32^2 + (-32)^2}$

$= \sqrt{32^2 + 32^2}$

$= \sqrt{2 \times 32^2}$

$= 32\sqrt{2}$

4 a $xy = 32 \Rightarrow y = 32x^{-1}$

$\frac{dy}{dx} = -32x^{-2}$

At $x = -2$

$\frac{dy}{dx} = -32(-2)^{-2}$

$= -8$

At $x = -2$ the gradient of the tangent is -8 so the gradient of the normal is $\frac{1}{8}$

Find the equation of the normal using

$y - y_1 = m(x - x_1)$ with $m = \frac{1}{8}$ at $(-2, -16)$

gives:

$y + 16 = \frac{1}{8}(x + 2)$

$x - 8y - 126 = 0$

b To find the coordinates of B, substitute $y = \frac{32}{x}$ into $x - 8y - 126 = 0$

$x - 8\left(\frac{32}{x}\right) - 126 = 0$

$x^2 - 126x - 256 = 0$

$(x + 2)(x - 128) = 0$

$x = -2$ or $x = 128$

When $x = 128$, $y = \frac{1}{4}$

So B has coordinates $\left(128, \frac{1}{4}\right)$

5 a $P(4, 12)$ and $Q(-8, -6)$ lie on $xy = 48$

The gradient of PQ is:

$m = \frac{y_P - y_Q}{x_P - x_Q}$

$= \frac{12 - (-6)}{4 - (-8)}$

$= \frac{3}{2}$

Find the equation of the line using

$y - y_1 = m(x - x_1)$ with $m = \frac{3}{2}$ at $(-8, -6)$

gives:

$y + 6 = \frac{3}{2}(x + 8)$

$3x - 2y + 12 = 0$ as required

5 b Since the normal at A is parallel to PQ it has a gradient of $\frac{3}{2}$ therefore the tangent at A has a gradient of $-\frac{2}{3}$

$xy = 48 \Rightarrow y = 48x^{-1}$

$\frac{dy}{dx} = -48x^{-2}$

Since $\frac{dy}{dx}$ represents the gradient

$-48x^{-2} = -\frac{2}{3}$

$\frac{48}{x^2} = \frac{2}{3}$

$x^2 = 72$

$x = \pm 6\sqrt{2}$

When $x = 6\sqrt{2}$, $y = 4\sqrt{2}$ and when $x = -6\sqrt{2}$, $y = -4\sqrt{2}$

So the possible coordinates of A are $(6\sqrt{2}, 4\sqrt{2})$ and $(-6\sqrt{2}, -4\sqrt{2})$

6 a $x = \sqrt{3}t,\ y = \frac{\sqrt{3}}{t},\ t \in \mathbb{R},\ t \neq 0$

$x = \sqrt{3}t \Rightarrow t = \frac{x}{\sqrt{3}}$

Substituting $t = \frac{x}{\sqrt{3}}$ into $y = \frac{\sqrt{3}}{t}$ gives:

$y = \frac{\sqrt{3}}{\left(\frac{x}{\sqrt{3}}\right)}$

$= \frac{3}{x}$

$xy = 3$

6 b $xy = 3 \Rightarrow y = 3x^{-1}$

$\frac{dy}{dx} = -3x^{-2}$

When $x = 2\sqrt{3}$

$\frac{dy}{dx} = -3\left(2\sqrt{3}\right)^{-2}$

$= -\frac{1}{4}$

At $x = 2\sqrt{3}$ the gradient of the tangent is $-\frac{1}{4}$ so the gradient of the normal is 4

At $x = 2\sqrt{3}$, $y = \frac{\sqrt{3}}{2}$

Find the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = 4$ at $\left(2\sqrt{3}, \frac{\sqrt{3}}{2}\right)$ gives:

$y - \frac{\sqrt{3}}{2} = 4\left(x - 2\sqrt{3}\right)$

$2y - \sqrt{3} = 8x - 16\sqrt{3}$

$8x - 2y - 15\sqrt{3} = 0$

c Substituting $y = \frac{3}{x}$ into $8x - 2y - 15\sqrt{3} = 0$ gives:

$8x - 2\left(\frac{3}{x}\right) - 15\sqrt{3} = 0$

$8x^2 - 15\sqrt{3}x - 6 = 0$

$x = \frac{15\sqrt{3} \pm \sqrt{\left(-15\sqrt{3}\right)^2 - 4(8)(-6)}}{2(8)}$

$= \frac{15\sqrt{3} \pm 17\sqrt{3}}{16}$

$x = 2\sqrt{3}$ or $x = -\frac{\sqrt{3}}{8}$

When $x = -\frac{\sqrt{3}}{8}$, $y = -8\sqrt{3}$

So Q has coordinates $\left(-\frac{\sqrt{3}}{8}, -8\sqrt{3}\right)$

7 a $P(4t^2, 8t)$ lies on $y^2 = 16x$ and on $xy = 4$

Substituting $x = 4t^2$ and $y = 8t$ into $xy = 4$ gives:

$$8t(4t^2) = 4$$
$$32t^3 = 4$$
$$t^3 = \frac{1}{8}$$
$$t = \frac{1}{2}$$

P has coordinates (1, 4)

b $xy = 4 \Rightarrow y = 4x^{-1}$

$$\frac{dy}{dx} = -4x^{-2}$$

When $x = 1$

$$\frac{dy}{dx} = -4(1)^{-2}$$
$$= -4$$

At $x = 1$ the gradient of the tangent is -4

so the gradient of the normal is $\frac{1}{4}$

Find the equation of the normal using $y - y_1 = m(x - x_1)$ with $m = \frac{1}{4}$ at (1, 4)

gives:

$$y - 4 = \frac{1}{4}(x - 1)$$
$$x - 4y + 15 = 0$$

The normal meets the x-axis at N where $y = 0$ so

$$x - 4(0) + 15 = 0$$
$$x = -15$$

So N has coordinates $(-15, 0)$

7 c $y^2 = 16x \Rightarrow y = \pm 4x^{\frac{1}{2}}$

At (1, 4), $y = 4x^{\frac{1}{2}}$ so $\frac{dy}{dx} = 2x^{-\frac{1}{2}}$

$$\frac{dy}{dx} = 2(1)^{-\frac{1}{2}}$$
$$= 2$$

Find the equation of the tangent using $y - y_1 = m(x - x_1)$ with $m = 2$ at (1, 4)

gives:

$$y - 4 = 2(x - 1)$$
$$2x - y + 2 = 0$$

The tangent meets the x-axis at T where $y = 0$ so

$$2x - (0) + 2 = 0$$
$$x = -1$$

So T has coordinates $(-1, 0)$

d $$\text{Area}_{NPT} = \frac{1}{2}(14)(4)$$
$$= 28$$