Exercise 4C

1 $y^2 = 3x$ and $y = 2x - 3$

Substituting for y gives:

$$(2x-3)^2 = 3x$$
$$4x^2 - 12x + 9 = 3x$$
$$4x^2 - 15x + 9 = 0$$
$$(4x-3)(x-3) = 0$$
$$x = \frac{3}{4} \text{ or } x = 3$$

When $x = \frac{3}{4}$, $y = -\frac{3}{2}$ and when $x = 3$, $y = 3$

Therefore the points P and Q are $\left(\frac{3}{4}, -\frac{3}{2}\right)$ and (3, 3)

2 $y^2 = 32x$ and $y = x + 6$

Substituting for y gives:

$$(x+6)^2 = 32x$$
$$x^2 + 12x + 36 = 32x$$
$$x^2 - 20x + 36 = 0$$
$$(x-2)(x-18) = 0$$
$$x = 2 \text{ or } x = 18$$

When $x = 2$, $y = 8$ and when $x = 18$, $y = 24$

Therefore the points A and B are $(2, 8)$ and $(18, 24)$

$$|AB| = \sqrt{(18-2)^2 + (24-8)^2}$$
$$= \sqrt{16^2 + 16^2}$$
$$= \sqrt{2 \times 16^2}$$
$$= 16\sqrt{2}$$

3 $y^2 = 10x$ and $y = x - 20$

Substituting for y gives:

$$(x-20)^2 = 10x$$
$$x^2 - 40x + 400 = 10x$$
$$x^2 - 50x + 400 = 0$$
$$(x-10)(x-40) = 0$$
$$x = 10 \text{ or } x = 40$$

When $x = 10$, $y = -10$ and when $x = 40$, $y = 20$

Therefore the points A and B are $(10, -10)$ and $(40, 20)$

The midpoint of AB is:

$$M = \left(\frac{10+40}{2}, \frac{-10+20}{2}\right)$$
$$= (25, 5)$$

4 a $x = 6t^2$, $y = 12t \Rightarrow t = \frac{y}{12}$

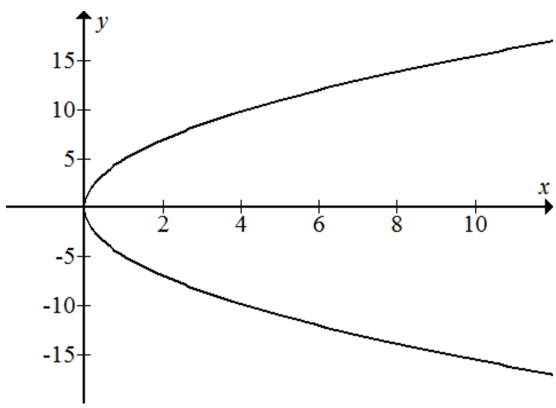
Substituting $t = \frac{y}{12}$ into $x = 6t^2$ gives:

$$x = 6\left(\frac{y}{12}\right)^2$$
$$= \frac{6y^2}{144}$$
$$y^2 = 24x$$

b The general equation of a parabola is $y^2 = 4ax$ with focus at $(a, 0)$ and directrix at $x + a = 0$

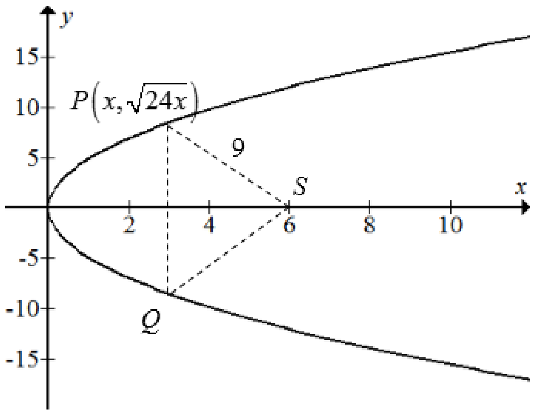
Since $y^2 = 24x$, $a = 6$, the focus is at (6, 0) and the directrix is at $x + 6 = 0$

c



4 d 9 units

e The coordinates of P are $\left(x, \sqrt{24x}\right)$



Since $SP = 9$,

$$\sqrt{(x-6)^2 + \left(\sqrt{24x} - 0\right)^2} = 9$$

$$x^2 - 12x + 36 + 24x = 81$$

$$x^2 + 12x - 45 = 0$$

$$(x-3)(x+15) = 0$$

$x = 3$ or $x = -15$

$x \neq -15$ therefore the coordinates of P are $\left(3, 6\sqrt{2}\right)$

and by symmetry the coordinates of Q are $\left(3, -6\sqrt{2}\right)$

so PQ has length $12\sqrt{2}$

f Area of triangle $PQS = \frac{1}{2} \times 12\sqrt{2} \times 3$

$= 18\sqrt{2}$

5 a $y^2 = 4ax$ and $\left(\frac{5}{4}t^2, \frac{5}{2}t\right)$ is a general point on the curve

Substituting $\left(\frac{5}{4}t^2, \frac{5}{2}t\right)$ into $y^2 = 4ax$ gives:

$$\left(\frac{5}{2}t\right)^2 = 4a\left(\frac{5}{4}t^2\right)$$

$$\frac{25}{4}t^2 = \frac{20}{4}at^2$$

$$a = \frac{5}{4}$$

Therefore the Cartesian equation of the curve is $y^2 = 5x$

b When $y = 5$, $x = 5$, therefore P has coordinates (5, 5)

c When $y^2 = 5x$ the directrix is at $x + \frac{5}{4} = 0$

Therefore Q has coordinates $\left(-\frac{5}{4}, 3\right)$

d The gradient of the line l is:

$$m = \frac{y_Q - y_P}{x_Q - x_P}$$

$$= \frac{3-5}{-\frac{5}{4} - 5}$$

$$= \frac{8}{25}$$

Finding the equation of l using $y - y_1 = m(x - x_1)$ with $m = \frac{8}{25}$ at the point (5, 5) gives:

$$y - 5 = \frac{8}{25}(x - 5)$$

$$8x - 25y + 85 = 0$$

6 a $y^2 = 4x$

The general equation of a parabola is $y^2 = 4ax$ with focus at $(a, 0)$ and directrix at $x + a = 0$

Therefore for C, $a = 1$ and the focus, S, is at $(1, 0)$

b When $y = 4$, $x = 4$, therefore P has coordinates $(4, 4)$

c The gradient of the line l is:

$$m = \frac{y_Q - y_P}{x_Q - x_P} = \frac{0-4}{1-4} = \frac{4}{3}$$

Finding the equation of l using $y - y_1 = m(x - x_1)$ with $m = \frac{4}{3}$ at the point $(1, 0)$ gives:

$$y - 0 = \frac{4}{3}(x-1)$$

$$4x - 3y - 4 = 0$$

d The line $4x - 3y - 4 = 0$ meets $y^2 = 4x$

Rearranging gives $4x = 3y + 4$ and substituting into $y^2 = 4x$ gives:

$$(y)^2 = 3y + 4$$

$$y^2 - 3y - 4 = 0$$

$$(y-4)(y+1) = 0$$

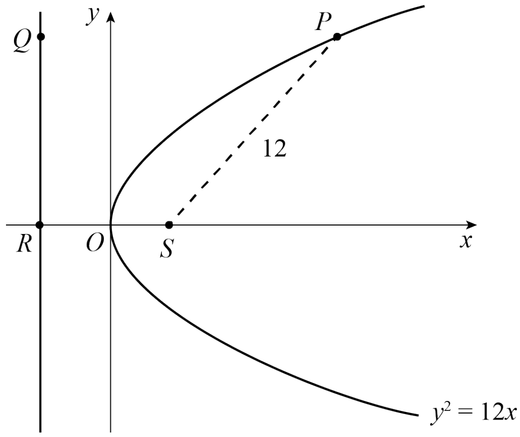
$$y = 4 \text{ or } y = -1$$

When $y = -1$, $x = \frac{1}{4}$, therefore Q has coordinates $\left(\frac{1}{4}, -1\right)$

e Q has coordinates $\left(\frac{1}{4}, -1\right)$ and the directrix lies at $x = -1$

Therefore the distance from the directrix to Q is $\frac{5}{4}$

7 a $y^2 = 12x$



The general equation of a parabola is $y^2 = 4ax$ with focus at $(a, 0)$ and directrix at $x + a = 0$

Therefore for C, $a = 3$, the focus is at $(3, 0)$ and the directrix is at $x + 3 = 0$

So S has coordinates $(3, 0)$ and R has coordinates $(-3, 0)$

b For all points on a parabola, the distance from the focus is equal to the distance from the directrix.

Therefore, $SP = QP = 12$

Thus $x + 3 = 12$

$$x = 9$$

When $x = 9$, $y = 6\sqrt{3}$ so P has coordinates $(9, 6\sqrt{3})$ and Q has coordinates $(-3, 6\sqrt{3})$

c The area of a trapezium is given by:

$$A = \frac{1}{2}(a+b)h$$

$$A_{PQRS} = \frac{1}{2}(6+12)(6\sqrt{3}) = 54\sqrt{3}$$

8 a $P(16, 8)$ and $Q(4, b)$ lie on $y^2 = 4ax$

Since $(16, 8)$ lies on $y^2 = 4ax$

$8^2 = 4(16)a$

$a = 1$

So the parabola is $y^2 = 4x$

and since $(4, b)$ lies on $y^2 = 4x$

$b^2 = 16$

$b = \pm 4$

Since $b < 0$, $b = -4$ and Q has coordinates $(4, -4)$

b P and Q lie on the line l so the gradient of l is:

$$m = \frac{y_Q - y_P}{x_Q - x_P}$$

$$= \frac{-4-8}{4-16}$$

$$= 1$$

Finding the equation of l using $y - y_1 = m(x - x_1)$ with $m = 1$ and $(4, -4)$ gives:

$$y + 4 = 1(x - 4)$$

$$y = x - 8$$

c To find the midpoint of PQ use:

$$\left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right)$$

So R has coordinates

$$\left(\frac{16+4}{2}, \frac{8-4}{2}\right) = (10, 2)$$

d Since n is perpendicular to l it has a gradient of -1

Finding the equation of n using $y - y_1 = m(x - x_1)$ with $m = -1$ and $(10, 2)$ gives:

$$y - 2 = -1(x - 10)$$

$$y = -x + 12$$

8 e The line $y = -x + 12$ meets the parabola at two points.

Substituting $y = -x + 12$ into $y^2 = 4x$ gives:

$$(-x+12)^2 = 4x$$

$$x^2 - 24x + 144 = 4x$$

$$x^2 - 28x + 144 = 0$$

$$x = \frac{28 \pm \sqrt{(-28)^2 - 4(1)(144)}}{2(1)}$$

$$= \frac{28 \pm \sqrt{208}}{2}$$

$$x = 14 \pm 2\sqrt{13}$$

So $\lambda = 14$ and $\mu = 2$