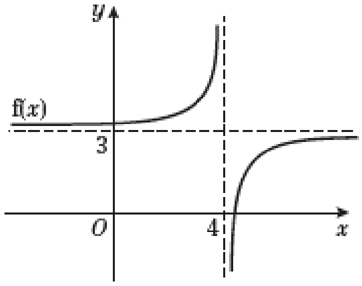
Chapter review 3

1 a $f(x) = \frac{1}{4-x} + 3$

$f(3.9) = \frac{1}{0.1} + 3 = 13$

$f(4.1) = -\frac{1}{0.1} + 3 = -7$

b There is an asymptote at $x = 4$ which causes the change of sign, not a root.



c $f(x) = 0 \Rightarrow \frac{1}{4-x} + 3 = 0$

$\frac{1}{x-4} = 3$

$1 = 3x - 12 \Rightarrow x = \frac{13}{3}$

So $\alpha = \frac{13}{3}$

2 $x^3 - 2x + 2 = 0$

Let $f(x) = x^3 - 2x + 2$

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
−1	3	−2	−2	−1.5	1.625
−1.5	1.625	−2	−2	−1.75	0.140625
−1.75	0.140625	−2	−2	−1.875	−1.84179…
−1.75	0.140625	−1.875	−1.84179…	−1.8125	−0.32934…
−1.75	0.140625	−1.8125	−0.32934…	−1.78125	−0.08914…

Two successive approximations give $x = -1.8$, accurate to 1 d.p.

3 $x^3 - 12x - 7.2 = 0$

Let $f(x) = x^3 - 12x - 7.2$

$f(-4) = -23.2$

$f(-2) = 8.8$

$f(0) = -7.2$

$f(3) = -16.2$

$f(4) = 8.8$

There is a sign change and hence a root in each of the intervals [−4, −2], [−2, 0] and [3, 4].
The equation therefore has one positive root and two negative roots.

$f'(x) = 3x^2 - 12$

The positive root lies in the interval [3, 4].
Using $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{-16.2}{15}$$

$$= 4.08$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 4.08 - \frac{11.757312}{37.9392}$$

$$= 3.7701012$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 3.7701012 - \frac{1.1457338}{30.6409892}$$

$$= 3.7327090$$

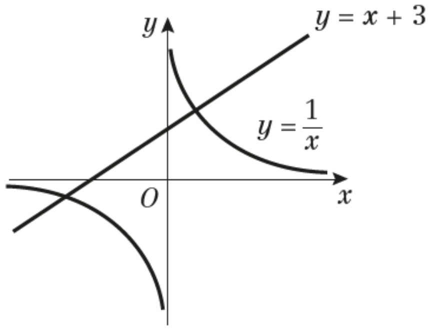
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 3.7327090 - \frac{0.0157613}{29.7993494}$$

$$= 3.7321800$$

Two successive approximations give $x = 3.73$ (3 s.f.)

4 a



b The line meets the curve at two points, so there are two values of x that satisfy the equation $\frac{1}{x} = x + 3$.

So $\frac{1}{x} = x + 3$ has two roots.

c $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$

Let f(x) = $x + 3 - \frac{1}{x}$

$f(0.30) = (0.30) + 3 - \frac{1}{0.30} = -0.0333...$

$f(0.31) = (0.31) + 3 - \frac{1}{0.31} = 0.0841...$

$f(0.30) < 0$ and $f(0.31) > 0$ so there is a change of sign, which implies there is a root between $x = 0.30$ and $x = 0.31$.

d $\frac{1}{x} = x + 3$

$\frac{1}{x} \times x = x \times x + 3 \times x$ Multiply by x.

$1 = x^2 + 3x$

So $x^2 + 3x - 1 = 0$

e Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1, b = 3, c = -1$

$$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{9+4}}{2} = \frac{-3 \pm \sqrt{13}}{2}$$

So $x = \frac{-3 + \sqrt{13}}{2} = 0.3027...$

The positive root is 0.303 to 3 d.p.

5 **a** $g(x) = x^3 - 7x^2 + 2x + 4$

$g'(x) = 3x^2 - 14x + 2$

b Using $x_0 = 6.6$,

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$$

$$= 6.6 - \frac{g(6.6)}{g'(6.6)}$$

$$= 6.6 - \frac{6.6^3 - 7(6.6^2) + 2(6.6) + 4}{3(6.6^2) - 14(6.6) + 2}$$

$= 6.606$ correct to 3 d.p.

c $g(1) = 0 \Rightarrow x - 1$ is a factor of $g(x)$

$g(x) = (x-1)(x^2 - 6x - 4)$

$(x-1)(x^2 - 6x - 4) = 0$

Other two roots of $g(x)$ are given by

$$\frac{6 \pm \sqrt{36+16}}{2} = \frac{6 \pm \sqrt{52}}{2} = 3 \pm \sqrt{13}$$

d Percentage error:

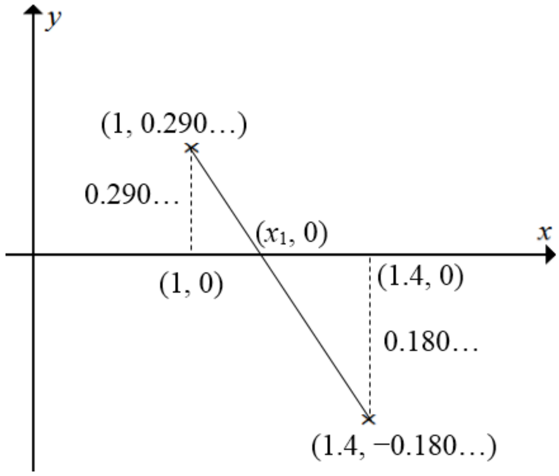
$$\frac{6.606 - \left(3 + \sqrt{13}\right)}{3 + \sqrt{13}} \times 100 = 0.007\%$$

6 $\cos x = \frac{1}{4}x \Rightarrow \cos x - \frac{1}{4}x = 0$

Let $f(x) = \cos x - \frac{1}{4}x$

$f(1.0) = 0.290...$

$f(1.4) = -0.180...$



By similar triangles:

$$\frac{1.4 - x_1}{x_1 - 1} = \frac{0.180...}{0.290...}$$

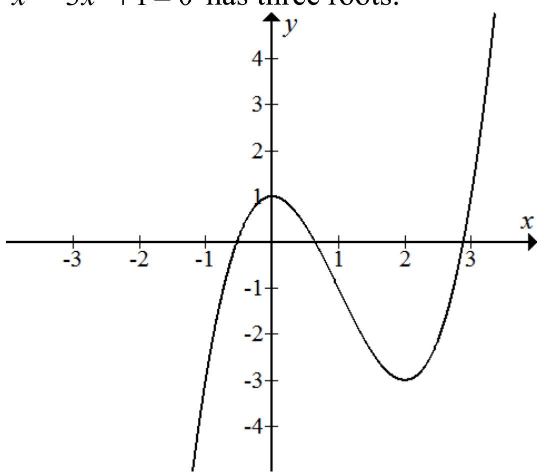
$0.290...(1.4 - x_1) = 0.180...(x_1 - 1)$

$0.470...x_1 = 0.586...$

$x_1 = 1.25$ (3 s.f.)

7 $f(x) = x^3 - 3x^2 + 1$

a A sketch of the graph shows that $x^3 - 3x^2 + 1 = 0$ has three roots.



7 b $f(x) = x^3 - 3x^2 + 1$

$f'(x) = 3x^2 - 6x$

Using $x_0 = -0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= -0.5 - \frac{0.125}{3.75}$$
$$= -0.5333333$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= -0.5333333 - \frac{0.0050370}{4.0533333}$$
$$= -0.5345760$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= -0.5345760 - \frac{-0.0100810}{4.0647705}$$
$$= -0.5320959$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= -0.5320959 - \frac{-0.0000283}{4.0419535}$$
$$= -0.5320890$$

So there is a root at $x = -0.532$ correct to 3 decimal places.

Using $x_0 = 0.6$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0.6 - \frac{0.136}{-2.52}$$
$$= 0.6539682$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.6539682 - \frac{-0.0033379}{-2.6407861}$$
$$= 0.6527042$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.6527042 - \frac{-0.0000015}{-2.6381569}$$
$$= 0.6527036$$

So there is a root at $x = 0.653$ correct to 3 decimal places.

Using $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 3 - \frac{1}{9}$$
$$= 2.8888889$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.8888889 - \frac{0.0727023}{7.7037037}$$
$$= 2.8794516$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.8794516 - \frac{0.0005040}{7.5970150}$$
$$= 2.8793853$$

So there is a root at $x = 2.879$ correct to 3 decimal places.

Challenge

a $f(y) = y^2 - 6y + 7$

$f(4) = -1$

$f(5) = 2$

There is a sign change so there is a root between $y = 4$ and $y = 5$.

$f(1) = 2$

$f(2) = -1$

There is a sign change so there is a root between $y = 1$ and $y = 2$.

b $f(y) = y^2 - 6y + 7$

$f'(y) = 2y - 6$

Using $y_0 = 5$

$$y_1 = y_0 - \frac{f(y_0)}{f'(y_0)}$$

$$= 5 - \frac{2}{4}$$

$$= 4.5$$

$$y_2 = y_1 - \frac{f(y_1)}{f'(y_1)}$$

$$= 4.5 - \frac{0.25}{3}$$

$$= 4.4164164$$

$$y_3 = y_2 - \frac{f(y_2)}{f'(y_2)}$$

$$= 4.4164164 - \frac{0.0069444}{2.8328328}$$

$$= 4.4139650$$

$$y_4 = y_3 - \frac{f(y_3)}{f'(y_3)}$$

$$= 4.4139650 - \frac{-0.0007030}{2.8279300}$$

$$= 4.4142136$$

$$y_5 = y_4 - \frac{f(y_4)}{f'(y_4)}$$

$$= 4.4142136 - \frac{0.0000001}{2.8284272}$$

$$= 4.4142136$$

So $y = 4.4142$ (5 s.f.) as both y_4 and y_5 give $y = 4.4142$ correct to 5 significant figures.