Exercise 2B

1 $x^2+5x+2=0$ has roots α and β

$a=1$, $b=5$ and $c=2$

$\alpha+\beta=-\frac{b}{a}=-5$ and $\alpha\beta=\frac{c}{a}=2$

a The sum of the roots is

$$(2\alpha+1)+(2\beta+1)=2(\alpha+\beta)+2=2(-5)+2=-8$$

The product of the roots is

$$(2\alpha+1)(2\beta+1)=4\alpha\beta+2\alpha+2\beta+1=4\alpha\beta+2(\alpha+\beta)+1=4(2)+2(-5)+1=-1$$

So $x^2+8x-1=0$

b The sum of the roots is

$$\alpha\beta+\alpha^2\beta^2=\alpha\beta(1+\alpha\beta)=2(1+2)=6$$

The product of the roots is

$$\alpha\beta(\alpha^2\beta^2)=\alpha^3\beta^3=(\alpha\beta)^3=2^3=8$$

So $x^2-6x+8=0$

2 $3x^2-2x+3=0$ has roots α and β

$a=3$, $b=-2$ and $c=3$

$\alpha+\beta=-\frac{b}{a}=\frac{2}{3}$ and $\alpha\beta=\frac{c}{a}=1$

2 a The sum of the roots is

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}=\frac{\left(\frac{2}{3}\right)^2-2(1)}{1^2}=-\frac{14}{9}$$

The product of the roots is

$$\frac{1}{\alpha^2}\left(\frac{1}{\beta^2}\right)=\frac{1}{(\alpha\beta)^2}=1$$

So $x^2+\frac{14}{9}x+1=0$

$9x^2+14x+9=0$

b The sum of the roots is

$$\frac{\beta}{\alpha^2}+\frac{\alpha}{\beta^2}=\frac{\alpha^3+\beta^3}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^3-(3\alpha^2\beta+3\alpha\beta^2)}{(\alpha\beta)^2}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^2}=\frac{\left(\frac{2}{3}\right)^3-3(1)\left(\frac{2}{3}\right)}{1^2}=-\frac{46}{27}$$

The product of the roots is

$$\frac{\beta}{\alpha^2}\left(\frac{\alpha}{\beta^2}\right)=\frac{\alpha\beta}{(\alpha\beta)^2}=1$$

So $x^2+\frac{46}{27}x+1=0$

$27x^2+46x+27=0$

3 $3x^2+7x+6=0$ has roots α and β

$a=3,\ b=7$ and $c=6$

$$\alpha+\beta=-\frac{b}{a}=-\frac{7}{3} \quad \text{and} \quad \alpha\beta=\frac{c}{a}=2$$

a The sum of the roots is

$$\begin{aligned}\alpha^2+\beta+\alpha+\beta^2&=\alpha^2+\beta^2+\alpha+\beta\\&=(\alpha+\beta)^2-2\alpha\beta+(\alpha+\beta)\\&=\left(-\frac{7}{3}\right)^2-2(2)+\left(-\frac{7}{3}\right)\\&=-\frac{8}{9}\end{aligned}$$

The product of the roots is

$$\begin{aligned}&(\alpha^2+\beta)(\alpha+\beta^2)\\&=\alpha^3+\alpha^2\beta^2+\alpha\beta+\beta^3\\&=(\alpha+\beta)^3-(3\alpha^2\beta+3\alpha\beta^2)+(\alpha\beta)^2+\alpha\beta\\&=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)+(\alpha\beta)^2+\alpha\beta\\&=\left(-\frac{7}{3}\right)^3-3(2)\left(-\frac{7}{3}\right)+2^2+2\\&=\frac{197}{27}\end{aligned}$$

So $x^2+\frac{8}{9}x+\frac{197}{27}=0$

$27x^2+24x+197=0$

b The sum of the roots is

$$\begin{aligned}\alpha^3+\beta^3&=(\alpha+\beta)^3-(3\alpha^2\beta+3\alpha\beta^2)\\&=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\\&=\left(-\frac{7}{3}\right)^3-3(2)\left(-\frac{7}{3}\right)\\&=\frac{35}{27}\end{aligned}$$

The product of the roots is

$$\begin{aligned}\alpha^3(\beta^3)&=(\alpha\beta)^3\\&=2^3\\&=8\end{aligned}$$

So $x^2-\frac{35}{27}x+8=0$

$27x^2-35x+216=0$

4 $6x^2-3x+4=0$ has roots α and β

$a=6,\ b=-3$ and $c=4$

$$\alpha+\beta=-\frac{b}{a}=\frac{1}{2} \quad \text{and} \quad \alpha\beta=\frac{c}{a}=\frac{2}{3}$$

a The sum of the roots is

$$\begin{aligned}\frac{1}{\alpha^3}+\frac{1}{\beta^3}&=\frac{\alpha^3+\beta^3}{\alpha^3\beta^3}\\&=\frac{(\alpha+\beta)^3-(3\alpha^2\beta+3\alpha\beta^2)}{(\alpha\beta)^3}\\&=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}\\&=\frac{\left(\frac{1}{2}\right)^3-3\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)}{\left(\frac{2}{3}\right)^3}\\&=-\frac{189}{64}\end{aligned}$$

The product of the roots is

$$\begin{aligned}\frac{1}{\alpha^3}\left(\frac{1}{\beta^3}\right)&=\frac{1}{(\alpha\beta)^3}\\&=\frac{1}{\left(\frac{2}{3}\right)^3}\\&=\frac{27}{8}\end{aligned}$$

So $x^2+\frac{189}{64}x+\frac{27}{8}=0$

$64x^2+189x+216=0$

4 b The sum of the roots is

$$\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta)$$
$$= \frac{2}{3}\left(\frac{1}{2}\right)$$
$$= \frac{1}{3}$$

The product of the roots is

$$\alpha^2\beta(\alpha\beta^2) = \alpha^3\beta^3$$
$$= (\alpha\beta)^3$$
$$= \left(\frac{2}{3}\right)^3$$
$$= \frac{8}{27}$$

So $x^2 - \frac{1}{3}x + \frac{8}{27} = 0$

$$27x^2 - 9x + 8 = 0$$