Chapter review

1 a $z_1 + z_2 = (8 - 3i) + (-2 + 4i)$
$= 6 + i$

b $3z_2 = 3(-2 + 4i)$
$= -6 + 12i$

c $6z_1 - z_2 = 6(8 - 3i) - (-2 + 4i)$
$= 48 - 18i + 2 - 4i$
$= 50 - 22i$

2 If $z^2 + bz + 14 = 0$ has no real roots then the discriminant < 0

$b^2 - 4(1)(14) < 0$

$b^2 < 56$

$-\sqrt{56} < b < \sqrt{56}$

$-2\sqrt{14} < b < 2\sqrt{14}$

3 $z^2 - 6z + 12 = 0$

Solve by completing the square:

$(z-3)^2 - 9 + 12 = 0$

$(z-3)^2 = -3$

$z - 3 = \pm i\sqrt{3}$

$z = 3 \pm i\sqrt{3}$

So $z_1 = 3 + i\sqrt{3}$ and $z_2 = 3 - i\sqrt{3}$

4 To expand $(1+2i)^5$ use the binomial expansion of $(a+b)^5$

$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

So

$(1+2i)^5 = (1)^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5$

$= 1 + 10i + 40i^2 + 80i^3 + 80i^4 + 32i^5$

Use the fact that $i^2 = -1, i^3 = -i, i^4 = 1$ and $i^5 = i$ to simplify

$(1+2i)^5 = 1 + 10i - 40 - 80i + 80 + 32i$

$= 41 - 38i$

5 $f(z) = z^2 - 6z + 10$

$f(3+i) = (3+i)^2 - 6(3+i) + 10$

$= 9 + 3i + 3i + i^2 - 18 - 6i + 10$

$= (9 - 1 - 18 + 10) + (3 + 3 - 6)i$

$= 0$

6 a If $z_1 = 4 + 2i$ then z_1^* is the complex conjugate. So $z_1^* = 4 - 2i$

b $z_1 z_2 = (4 + 2i)(-3 + i)$

$= -12 + 4i - 6i + 2i^2$

$= -14 - 2i$

c $\frac{z_1}{z_2} = \frac{4+2i}{-3+i}$

$= \frac{(4+2i)}{(-3+i)} \times \frac{(-3-i)}{(-3-i)}$

$= \frac{-12 - 4i - 6i - 2i^2}{9 + 3i - 3i - i^2}$

$= \frac{-10 - 10i}{10}$

$= -1 - i$

7 $(7-2i)^2 = 49 - 14i - 14i + 4i^2$

$= 45 - 28i$

$\frac{(7-2i)^2}{1+i\sqrt{3}} = \frac{(45-28i)}{(1+i\sqrt{3})} \times \frac{(1-i\sqrt{3})}{(1-i\sqrt{3})}$

$= \frac{45 - 45i\sqrt{3} - 28i + 28i^2\sqrt{3}}{1 - i\sqrt{3} + i\sqrt{3} - (\sqrt{3})^2 i^2}$

$= \frac{45 - 28\sqrt{3}}{4} + \left(\frac{-28 - 45\sqrt{3}}{4}\right)i$

8 $\frac{4-7i}{z}=3+i$

$z=\frac{4-7i}{3+i}$

$z=\frac{(4-7i)}{3+i}\times\frac{(3-i)}{(3-i)}$

$=\frac{12-4i-21i+7i^2}{9-3i+3i-i^2}$

$=\frac{5-25i}{10}$

$=\frac{1}{2}-\frac{5}{2}i$

9 $z=\frac{1}{2+i}$

$=\frac{1}{(2+i)}\times\frac{(2-i)}{(2-i)}$

$=\frac{2-i}{4-2i+2i-i^2}$

$z=\frac{2-i}{5}=\frac{2}{5}-\frac{1}{5}i$

a $z^2=\left(\frac{2-i}{5}\right)\left(\frac{2-i}{5}\right)$

$=\frac{4-2i-2i+i^2}{25}$

So $z^2=\frac{3}{25}-\frac{4}{25}i$

b $z-\frac{1}{z}=\left(\frac{2}{5}-\frac{1}{5}i\right)-\frac{1}{\left(\frac{1}{2+i}\right)}$

$=\left(\frac{2}{5}-\frac{1}{5}i\right)-(2+i)$

$=-\frac{8}{5}-\frac{6}{5}i$

10 If $z=a+bi$, then $z^*=a-bi$

So $\frac{z}{z^*}=\frac{a+bi}{a-bi}$

$=\frac{(a+bi)}{(a-bi)}\times\frac{(a+bi)}{(a+bi)}$

$=\frac{a^2+abi+abi+b^2i^2}{a^2+abi-abi-b^2i^2}$

$=\frac{a^2-b^2+2abi}{a^2+b^2}$

$=\left(\frac{a^2-b^2}{a^2+b^2}\right)+\left(\frac{2ab}{a^2+b^2}\right)i$

11 a $z=\frac{3+qi}{q-5i}$

Multiply by the complex conjugate

$z=\frac{(3+qi)}{(q-5i)}\times\frac{(q+5i)}{(q+5i)}$

$=\frac{3q+15i+q^2i+5qi^2}{q^2+5qi-5qi-25i^2}$

$=\left(\frac{-2q}{q^2+25}\right)+\left(\frac{q^2+15}{q^2+25}\right)i$

The real part of z is $\frac{1}{13}$, so

$\frac{-2q}{q^2+25}=\frac{1}{13}$

$-26q=q^2+25$

$q^2+26q+25=0$

$(q+25)(q+1)=0$

So $q=-1$ or $q=-25$

b If $q=-1$

$z=\frac{-2(-1)}{(-1)^2+25}+\frac{(-1)^2+15}{(-1)^2+25}i$

$=\frac{2}{26}+\frac{16}{26}i$

$=\frac{1}{13}+\frac{8}{13}i$

If $q=-25$

$z=\left(\frac{-2(-25)}{(-25)^2+25}\right)+\left(\frac{(-25)^2+15}{(-25)^2+25}\right)i$

$=\frac{50}{650}+\frac{640}{650}i$

$=\frac{1}{13}+\frac{64}{65}i$

12 $z+4\text{i}z^*=-3+18\text{i}$ (*)

Substitute $z=x+\text{i}y$ and $z^*=x-\text{i}y$ in (*)

$x+\text{i}y+4\text{i}(x-\text{i}y)=-3+18\text{i}$

$x+\text{i}y+4x\text{i}-4y\text{i}^2=-3+18\text{i}$

$(x+4y)+(4x+y)\text{i}=-3+18\text{i}$

Equate real parts and imaginary parts:

$x+4y=-3$ **(1)**

$4x+y=18$ **(2)**

Multiply **(2)** by 4:

$16x+4y=72$ **(3)**

Subtract **(1)** from **(3)**

$15x=75$, so $x=5$

Substitute $x=5$ into **(2)**

$4(5)+y=18$, so $y=-2$.

13 $\frac{z}{w}=\frac{9+6\text{i}}{2-3\text{i}}$

Multiply by the complex conjugate

$\frac{z}{w}=\frac{(9+6\text{i})}{2-3\text{i}}\times\frac{(2+3\text{i})}{2+3\text{i}}$

$=\frac{18+27\text{i}+12\text{i}+18\text{i}^2}{4+6\text{i}-6\text{i}-9\text{i}^2}$

$=\frac{39}{13}\text{i}$

$=3\text{i}$

14 $z=\frac{q+3\text{i}}{4+q\text{i}}$

Multiply by the complex conjugate

$z=\frac{(q+3\text{i})}{(4+q\text{i})}\times\frac{(4-q\text{i})}{(4-q\text{i})}$

$=\frac{4q-q^2\text{i}+12\text{i}-3q\text{i}^2}{16-4q\text{i}+4q\text{i}-q^2\text{i}^2}$

$=\frac{4q+3q}{16+q^2}+\frac{12-q^2}{16+q}\text{i}$

$=\frac{7q}{16+q^2}+\frac{12-q^2}{16+q^2}\text{i}$

15 a $z^2+5z+10=0$

Solve by completing the square.

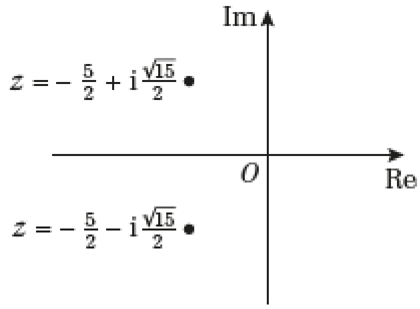
$\left(z+\frac{5}{2}\right)^2-\frac{25}{4}+10=0$

$\left(z+\frac{5}{2}\right)^2=-\frac{15}{4}$

$z+\frac{5}{2}=\pm\frac{\sqrt{15}}{2}\text{i}$

$z=-\frac{5}{2}\pm\frac{\sqrt{15}}{2}\text{i}$

b



16 a If $6-2\text{i}$ is a root, then the complex conjugate $6+2\text{i}$ is also a root.

b $(z-(6-2\text{i}))(z-(6+2\text{i}))$

$=z^2-(6+2\text{i})z-(6-2\text{i})z$
$+(6-2\text{i})(6+2\text{i})$

$=z^2-6z-2\text{i}z-6z+2\text{i}z$
$+36+12\text{i}-12\text{i}-4\text{i}^2$

$=z^2-12z+36+4$

$=z^2-12z+40$

The equation is $z^2-12z+40=0$

17 If $z = 4 - k\text{i}$ is a root, then the complex conjugate $4 + k\text{i}$ is also a root.

$$\left(z-(4-k\text{i})\right)\left(z-(4+k\text{i})\right)$$
$$= z^2 - (4+k\text{i})z - (4-k\text{i})z + (4-k\text{i})(4+k\text{i})$$
$$= z^2 - 4z - k\text{i}z - 4z + k\text{i}z + 16 + 4k\text{i} - 4k\text{i} - k^2\text{i}^2$$
$$= z^2 - 8z + 16 + k^2$$

So $z^2 - 8z + 16k^2 = z^2 - 2mz + 52$

Equate z coefficients:

$-8 = 2m$, so $m = 4$.

Equate constants:

$16 + k^2 = 52$, so $k^2 = 36 \Rightarrow k = \pm 6$

But $k > 0$, so $k = 6$.

18 If $2+\text{i}$ is a root of $\text{h}(z) = 0$ then the complex conjugate $2-\text{i}$ is also a root.

$$\text{h}(z) = \left(z-(2+\text{i})\right)\left(z-(2-\text{i})\right)$$
$$= z^2 - (2-\text{i})z - (2+\text{i})z + (2+\text{i})(2-\text{i})$$
$$= z^2 - 2z + \text{i}z - 2z - \text{i}z + 4 - 2\text{i} + 2\text{i} - \text{i}^2$$
$$= z^2 - 4z + 5$$

Divide to find the remaining factor

$$\begin{array}{r} z+4 \\ z^2-4z+5 \overline{\big)\, z^3+0z^2-11z+20} \\ z^3-4z^2+\ 5z \\ 4z^2-16z+20 \\ 4z^2-16z+20 \\ 0 \end{array}$$

So $\text{h}(z) = (z^2 - 4z + 5)(z+4) = 0$

has roots $2+\text{i}$, $2-\text{i}$ and -4

19 If $\text{f}(1+3\text{i}) = 0$, then $\text{f}(1-3\text{i}) = 0$.

So $\text{f}(z) = \left(z-(1+3\text{i})\right)\left(z-(1-3\text{i})\right)$
$$= z^2 - (1-3\text{i})z - (1+3\text{i})z + (1+3\text{i})(1-3\text{i})$$
$$= z^2 - z + 3\text{i}z - z - 3\text{i}z + 1 - 3\text{i} + 3\text{i} - 9\text{i}^2$$
$$= z^2 - 2z + 10$$

Divide to find the remaining factor

$$\begin{array}{r} z+2 \\ z^2-2z+10 \overline{\big)\, z^3+0z^2+\ 6z+20} \\ z^3-2z^2+10z \\ 2z^2-\ 4z+20 \\ 2z^2-\ 4z+20 \\ 0 \end{array}$$

So $z^3 + 6z + 20 = (z^2 - 2z + 10)(z+2)$

either $(z^2 - 2z + 10) = 0 \Rightarrow z = 1 \pm 3\text{i}$

(as these were the roots given at the beginning of the question)

or $z + 2 = 0 \Rightarrow z = -2$

So the roots of $z^3 + 6z + 20 = 0$ are $1+3\text{i}$, $1-3\text{i}$ and -2

20 a $\text{f}(z) = z^3 + 3z^2 + kz + 48$

If $\text{f}(4\text{i}) = 0$, then:

$$0 = \text{f}(4\text{i})$$
$$= (4\text{i})^3 + 3(4\text{i})^2 + k(4\text{i}) + 48$$
$$= 64\text{i}^3 + 48\text{i}^2 + 4k\text{i} + 48$$
$$= -64\text{i} - 48 + 4k\text{i} + 48$$
$$= (4k - 64)\text{i}$$

Hence $4k - 64 = 0 \Rightarrow k = 16$.

20 b If 4i is a root then -4i is also a root and $(z-4\text{i})(z-(-4\text{i}))$ is a quadratic factor of $\text{f}(z)$

Now $(z-4\text{i})(z-(-4\text{i})) = (z-4\text{i})(z+4\text{i})$

$$= z^2 + 4\text{i}z - 4\text{i}z - 16\text{i}^2$$
$$= z^2 + 16$$

Divide to find the remaining factor

$$\begin{array}{r} z+3 \\ z^2+0z+16\overline{\big)\,z^3+3z^3+16z+48} \\ z^3+0z^2+16z \\ 3z^2+\ 0z+48 \\ 3z^2+\ 0z+48 \\ 0 \end{array}$$

So $\text{f}(z) = (z^2+16)(z+3) = 0$ has two other roots which are -4i and -3

21 a If $\text{f}(-1+2\text{i}) = 0$, then $(z-(-1+2\text{i}))$ and $(z-(-1-2\text{i}))$ are factors of $\text{f}(z)$.

Hence

$$(z-(-1+2\text{i}))(z-(-1-2\text{i}))$$
$$= z^2 - (-1-2\text{i})z - (-1+2\text{i})z + (-1+2\text{i})(-1-2\text{i})$$
$$= z^2 + z + 2\text{i}z + z - 2\text{i}z + 1 + 2\text{i} - 2\text{i} - 4\text{i}^2$$

$= z^2 + 2z + 5$ is a quadratic factor of $\text{f}(z)$

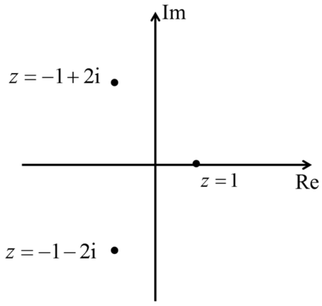
Use division to find the remaining factor.

$$\begin{array}{r} z-1 \\ z^2+2z+5\overline{\big)\,z^3+\ z^2+3z-5} \\ z^3+2z^2+5z \\ -z^2-2z-5 \\ -z^2-2z-5 \\ 0 \end{array}$$

So, $z^3 + z^2 + 3z - 5 = (z-1)(z^2+2z+5)$

So the solutions of $\text{f}(z) = 0$ are $-1+2\text{i}$, $-1-2\text{i}$ and 1.

21 b



c The vertices of the triangle are $A(-1,\ 2)$, $B(-1,-2)$ and $C(1,\ 0)$.

Use the distance formula to find the length of each side:

$$AB = \sqrt{(0)^2 + (4)^2} = 4$$
$$AC = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$
$$BC = \sqrt{(2)^2 + (2)^2} = \sqrt{8} = 2\sqrt{2}$$

Then $AC^2 + BC^2 = (2\sqrt{2})^2 + (2\sqrt{2})^2$

$$= 4^2$$
$$= AB^2$$

So by Pythagoras' theorem, ABC is a right-angled triangle which has a right-angle at C.

22 a $z^4 - z^3 - 16z^2 - 74z - 60$

$$= (z^2 - 5z - 6)(z^2 + bz + c)$$

Equate z^3 coefficients:

$b - 5 = -1$, so $b = 4$.

Equate constants:

$-6c = -60$, so $c = 10$.

22 b $0 = \mathrm{f}(z)$

$= (z^2 - 5z - 6)(z^2 + 4z + 10)$

$= (z - 6)(z + 1)(z^2 + 4z + 10)$

Either $z - 6 = 0 \Rightarrow z = 6$

or $z + 1 = 0 \Rightarrow z = -1$

or $z^2 + 4z + 10 = 0$

$(z + 2)^2 - 4 + 10 = 0$

$(z + 2)^2 = -6$

$z + 2 = \pm \mathrm{i}\sqrt{6}$

$z = -2 \pm \mathrm{i}\sqrt{6}$

So the roots of $\mathrm{f}(z) = 0$

are $6,\ -1,\ -2 + \mathrm{i}\sqrt{6}$ and $-2 - \mathrm{i}\sqrt{6}$.

23 If $\mathrm{g}(3 - 2\mathrm{i}) = 0$ then $\mathrm{g}(3 + 2\mathrm{i}) = 0$ and

$(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$ is a quadratic factor of $\mathrm{g}(z)$

Now $(z - (3 - 2\mathrm{i}))(z - (3 + 2\mathrm{i}))$

$= z^2 - (3 + 2\mathrm{i})z - (3 - 2\mathrm{i})z + (3 - 2\mathrm{i})(3 + 2\mathrm{i})$

$= z^2 - 3z - 2\mathrm{i}z - 3z + 2\mathrm{i}z + 9 + 6\mathrm{i} - 6\mathrm{i} - 4\mathrm{i}^2$

$= z^2 - 6z + 9 + 4$

$= z^2 - 6z + 13$

Divide to find the remaining factor

$$\begin{array}{r}
z^2 + 6 \\
z^2 - 6z + 13 \,\big)\, z^4 - 6z^3 + 19z^2 - 36z + 78 \\
z^4 - 6z^3 + 13z^2 \qquad\qquad \\
6z^2 - 36z + 78 \\
6z^2 - 36z + 78 \\
0
\end{array}$$

$0 = \mathrm{g}(z)$

$= (z^2 + 6)(z^2 - 6z + 13)$

Either $z^2 - 6z + 13 = 0$

$\Rightarrow z = 3 \pm 2\mathrm{i}$

(as these were the two original roots)

or $z^2 + 6 = 0$

$z^2 = -6$

$z = \pm \mathrm{i}\sqrt{6}$

23 (cont.)

So the roots of $\mathrm{g}(z) = 0$ are

$3 + 2\mathrm{i},\ 3 - 2\mathrm{i},\ \mathrm{i}\sqrt{6}$ and $-\mathrm{i}\sqrt{6}$.

24 a $\mathrm{f}(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

$0 = \mathrm{f}(4)$

$= 4^4 - 2(4)^3 - 5(4)^2 + 4p + 24$

$= 256 - 128 - 80 + 4p + 24$

$4p = -72$

$p = -18$

b If $\mathrm{f}(4) = 0$ then $(z - 4)$ is a factor of $\mathrm{f}(z)$

Divide to find the remaining factors

$$\begin{array}{r}
z^3 + 2z^3 + 3z - 6 \\
z - 4 \,\big)\, z^4 - 2z^3 - 5z^2 - 18z + 24 \\
z^4 - 4z^3 \qquad\qquad\qquad \\
2z^3 - 5z^2 \qquad\qquad \\
2z^3 - 8z^2 \qquad\qquad \\
3z^2 - 18z \qquad \\
3z^3 - 12z \qquad \\
-6z + 24 \\
-6z + 24 \\
0
\end{array}$$

So $0 = \mathrm{f}(z)$

$= (z - 4)(z^3 + 2z^2 + 3z - 6)$

Either $z - 4 = 0 \Rightarrow z = 4$

or $z^3 + 2z^2 + 3z - 6 = 0$ (*)

Now (*) is a cubic equation, so must have at least one real root (since complex roots always come in pairs).

Try small real values of z to find a factor of (*):

Let $z = 1$: $(1)^3 + 2(1)^2 + 3(1) - 6 = 0$

So $z = 1$ is a root of (*), and hence $z - 1$ is a factor.

So $z^3 + 2z^2 + 3z - 6 = (z - 1)(z^2 + bz + c)$

Equate z^2 terms:

$-1 + b = 2$, so $b = 3$

Equate constants:

$-6 = -c$, so $c = 6$

24 b (cont.)

Hence

$z^3+2z^2+3z-6=(z-1)(z^2+3z+6)$

Either $z=1$

or

$$(z^2+3z+6)=0$$

$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+6=0$$

$$\left(z+\frac{3}{2}\right)^2=-\frac{15}{4}$$

$$z+\frac{3}{2}=\pm\frac{\sqrt{15}}{2}\text{i}$$

$$z=-\frac{3}{2}\pm\frac{\sqrt{15}}{2}\text{i}$$

So the roots of $\text{f}(z)=0$ are

$1,\ 4,\ \left(-\frac{3}{2}+\frac{\sqrt{15}}{2}\text{i}\right)$ and $\left(-\frac{3}{2}-\frac{\sqrt{15}}{2}\text{i}\right)$

25 a $z=-1+4\text{i}$ is a solution implies that the complex conjugate $z=-1-4\text{i}$ is also a solution.

Hence

$(z-(-1+4\text{i}))(z-(-1-4\text{i}))$

$=z^2-(-1-4\text{i})z-(-1+4\text{i})z$

$\quad+(-1+4\text{i})(-1-4\text{i})$

$=z^2+z+4\text{i}z+z-4\text{i}z+1+4\text{i}-4\text{i}-16\text{i}^2$

$=z^2+2z+17$ is a quadratic factor of $\text{f}(z)$

Use long division to find the remaining factors.

$$\begin{array}{r|l} & z^2-3z+2 \\ \hline z^2+2z+17 & z^4-z^3+13z^2-47z+34 \\ & z^4+2z^3+17z^2 \\ & \quad -3z^3-4z^2-47z \\ & \quad -3z^3-6z^2-51z \\ & \qquad 2z^2+4z+34 \\ & \qquad 2z^2+4z+34 \\ & \qquad\qquad 0 \end{array}$$

So $z^4-z^3+13z^2-47z+34$

$\quad=(z^2+2z+17)(z^2-3z+2)$

Either $z^2+2z+17=0$

$\Rightarrow z=-1+4\text{i}$ or $z=-1-4\text{i}$

25 a (cont.)

Or $z^2-3z+2=0$

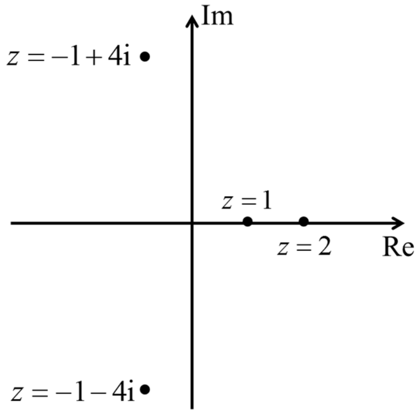
$(z-2)(z-1)=0$

$\Rightarrow z=1$ or $z=2$

So the roots of $\text{f}(z)=0$ are

$1,\ 2,\ -1+4\text{i}$ and $-1-4\text{i}$.

b



26 a $(4-3\text{i})x-(1+6\text{i})y-3=0$

$4x-3\text{i}x-y-6\text{i}y-3=0$

Equate real parts:

$4x-y-3=0 \quad (1)$

Equate imaginary parts:

$-3x-6y=0 \quad (2)$

From (2), $x=-2y \quad (3)$

Substitute $x=-2y$ into (1):

$4(-2y)-y-3=0$

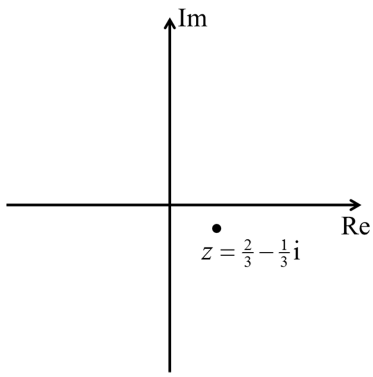
$-9y-3=0$

$y=-\frac{1}{3}$

Sub $y=-\frac{1}{3}$ into (3)

$x=-2\left(-\frac{1}{3}\right)=\frac{2}{3}$

b



26 c $|z|=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$

d $\tan\alpha=\frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}=\frac{1}{2}$

$\alpha=\tan^{-1}\left(\frac{1}{2}\right)=0.4636...$

As z is in the fourth quadrant,

$\arg z=-\alpha=-0.46$ radians (2 d.p.).

27 a $z=a+4\text{i}$

$z^2=(a+4\text{i})^2$

$=a^2+4a\text{i}+4a\text{i}+16\text{i}^2$

$=(a^2-16)+8a\text{i}$

$2z=2a+8\text{i}$

$z^2+2z=(a^2-16)+8a\text{i}+2a+8a\text{i}$

$=(a^2+2a-16)+(8a+8)\text{i}$

b If z^2+2z is real, then $\text{Im}(z^2+2z)=0$

So, $8a+8=0\Rightarrow a=-1$

c $z=a+4\text{i}$, so $z=-1+4\text{i}$

$|z|=\sqrt{(-1)^2+(-4)^2}=\sqrt{17}\approx 4.12$

$\tan\alpha=\frac{4}{1}$

$\alpha=\tan^{-1}4=1.3258...$

As z is in the second quadrant,

$\arg z=\pi-\alpha$.

$\arg z=\pi-1.3258...\approx 1.82$

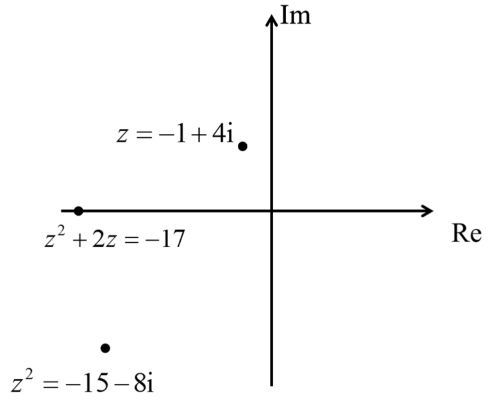
27 d Use $a=-1$ to find z, z^2 and z^2+2z

$z=a+4\text{i}=-1+4\text{i}$

$z^2=(a^2-16)+8a\text{i}=-15-8\text{i}$

$z^2+2z=(a^2+2a-16)+(8a+8)\text{i}=-17$

Show these points on an Argand diagram.



28 a $z=\frac{3+5\text{i}}{2-\text{i}}$

Multiply by the complex conjugate

$z=\frac{(3+5\text{i})}{(2-\text{i})}\times\frac{(2+\text{i})}{(2+\text{i})}$

$=\frac{6+3\text{i}+10\text{i}+5\text{i}^2}{4+2\text{i}-2\text{i}-\text{i}^2}$

$=\frac{1+13\text{i}}{5}$

$=\frac{1}{5}+\frac{13}{5}\text{i}$

$|z|=\sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{13}{5}\right)^2}=\sqrt{\frac{170}{25}}=\frac{1}{5}\sqrt{170}$

b $\tan\alpha=\frac{\left(\frac{13}{5}\right)}{\left(\frac{1}{5}\right)}=13$

$\alpha=\tan^{-1}13\approx 1.49$

As z is in the first quadrant,

$\arg z=\alpha\approx 1.49$ (2 d.p.)

29 a $z=1+2i$

$z^2=(1+2\text{i})^2$

$=1+2\text{i}+2\text{i}+4\text{i}^2$

$=-3+4i$

$z^2-z=(-3+4\text{i})-(1+2\text{i})=-4+2\text{i}$

$|z^2-z|=\sqrt{(-4)^2+(2)^2}=\sqrt{20}=2\sqrt{5}$

29 b $\tan\alpha = \frac{2}{4} = \frac{1}{2}$

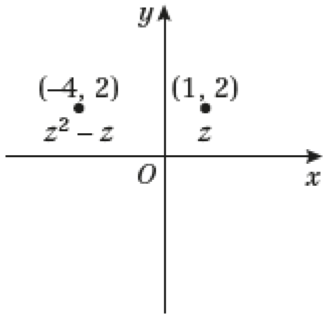
$\alpha = \tan^{-1}\frac{1}{2} = 0.46...$

As $(z^2 - z)$ is in the second quadrant,

$\arg(z^2 - z) = \pi - \alpha$

$\arg(z^2 - z) = \pi - 0.46... \approx 2.68$

c



30 a i $z = \frac{1}{2+\mathrm{i}} \times \frac{2-\mathrm{i}}{2-\mathrm{i}} = \frac{2-\mathrm{i}}{5}$

$= \frac{2}{5} - \frac{1}{5}\mathrm{i}$

$z^2 = \left(\frac{2}{5} - \frac{1}{5}\mathrm{i}\right)^2$

$= \frac{4}{25} - \frac{4}{25}\mathrm{i} + \left(\frac{1}{5}\mathrm{i}\right)^2$

$= \frac{4}{25} - \frac{4}{25}\mathrm{i} - \frac{1}{25}$

$= \frac{3}{25} - \frac{4}{25}\mathrm{i}$

ii $z - \frac{1}{z} = -\frac{2}{5} - \frac{1}{5}\mathrm{i} - (2+\mathrm{i})$

$= \frac{2}{5} - \frac{1}{5}\mathrm{i} - 2 - \mathrm{i}$

$= -\frac{8}{5} - \frac{6}{5}\mathrm{i}$

b

$|z^2|^2 = \left(\frac{3}{25}\right)^2 + \left(-\frac{4}{25}\right)^2$

$= \frac{9}{625} + \frac{16}{625} = \frac{25}{625} = \frac{1}{25}$

$|z^2| = \frac{1}{5}$

c $z - \frac{1}{z} = -\frac{8}{5} - \frac{6}{5}\mathrm{i}$

$\tan\alpha = \frac{\left(\frac{6}{5}\right)}{\left(\frac{8}{5}\right)} = \frac{6}{8} = \frac{3}{4}$

$\alpha = \tan^{-1}\frac{3}{4} = 0.6435...$

As z is in the third quadrant,

$\arg z = -(\pi - \alpha)$

$\arg z = -(\pi - 0.6435...) \approx -2.50$

31 a $z = \frac{a+3\mathrm{i}}{2+a\mathrm{i}} = \frac{a+3\mathrm{i}}{2+a\mathrm{i}} \times \frac{2-a\mathrm{i}}{2-a\mathrm{i}}$

$= \frac{2a - a^2\mathrm{i} + 6\mathrm{i} + 3a}{4+a^2}$

$= \frac{5a}{4+a^2} + \frac{6-a^2}{4+a^2}\mathrm{i} \qquad (*)$

Substitute $a = 4$ into (*)

$z = \frac{20}{20} + \frac{-10}{20}\mathrm{i} = 1 - \frac{1}{2}\mathrm{i}$

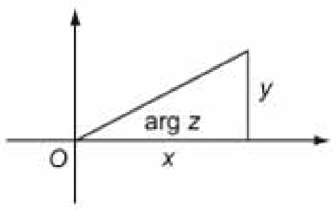
$|z|^2 = 1^2 + \left(-\frac{1}{2}\right)^2 = \frac{5}{4}$

$|z| = \frac{\sqrt{5}}{2}$

31 b Now

$\arg z = \frac{\pi}{4} \Rightarrow \tan(\arg z) = \tan\left(\frac{\pi}{4}\right) = 1$

If $z = x + \mathrm{i}y$, then $\tan(\arg z) = \frac{y}{x}$.



Hence

$1 = \tan(\arg z)$

$= \frac{\left(\frac{6-a^2}{4+a^2}\right)}{\left(\frac{5a}{4+a^2}\right)}$

$= \frac{6-a^2}{5a}$

$5a = 6 - a^2$

$a^2 + 5a - 6 = 0$

$(a+6)(a-1) = 0$

If $a = -6$, substituting into the result (*) in part a gives

$z = \frac{30}{40} - \frac{30}{40}\mathrm{i} = \frac{3}{4} - \frac{3}{4}\mathrm{i}$

31 b This is in the third quadrant and has a negative argument $\left(-\frac{3\pi}{4}\right)$, so $a = -6$ does not give $\arg z = \frac{\pi}{4}$

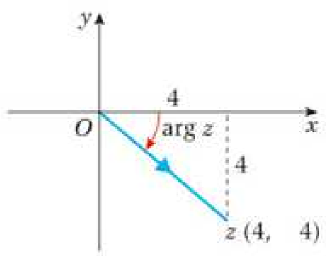
If $a = 1$, substituting into the result (*) in part a gives

$z = \frac{5}{5} + \frac{5}{5}\text{i} = 1 + \text{i}$

This is in the first quadrant and does have an argument $\frac{\pi}{4}$.

Therefore $a = 1$ is the only possible value of a.

32 $4 - 4\text{i}$



modulus $r = \sqrt{4^2 + (-4)^2} = \sqrt{16 + 16}$

$= \sqrt{32} = \sqrt{16}\sqrt{2} = 4\sqrt{2}$

argument $= \theta = -\tan^{-1}\left(\frac{4}{4}\right) = -\frac{\pi}{4}$

$$4 - 4\text{i} = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

$$= 4\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) - \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

Challenge

a Suppose, for a contradiction, that a cubic equation has repeated non-real roots. This repeated root gives us two roots of the cubic.
The complex conjugate of each repeated root would also be a root of the cubic.
This means we have found four roots of the cubic.
However, this is a contradiction, as a cubic has exactly three roots in total.
Therefore, a cubic equation cannot have repeated non-real roots.

b Any answer of the form $(z + a\text{i})^2(z - a\text{i})^2$ will work.

$(z + a\text{i})^2(z - a\text{i})^2$

$= (z^2 + 2za\text{i} + a^2\text{i}^2)(z^2 - 2za\text{i} + a^2\text{i}^2)$

$= z^4 - 2a\text{i}z^3 + a^2\text{i}^2z^2 + 2a\text{i}z^3 - 4a^2\text{i}^2z^2$
$\quad + 2a^3\text{i}^3z + a^2\text{i}^2z^2 - 2a^3\text{i}^3z + a^4\text{i}^4$

$= z^4 - 2az^3\text{i} - a^2z^2 + 2az^3\text{i} + 4a^2z^2$
$\quad - 2a^3z\text{i} - a^2z^2 + 2a^3z\text{i} + a^4$

$= z^4 + 2a^2z^2 + a^4$

This is a quadratic equation with real coefficients. Its roots are non-real and repeated.