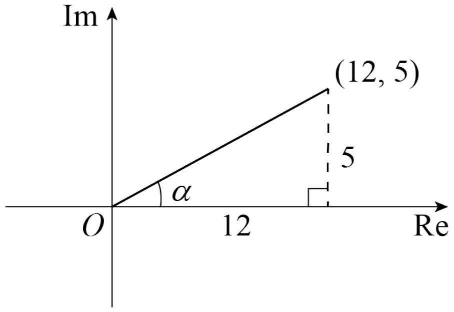
Exercise 1F

1 a $z = 12 + 5\text{i}$

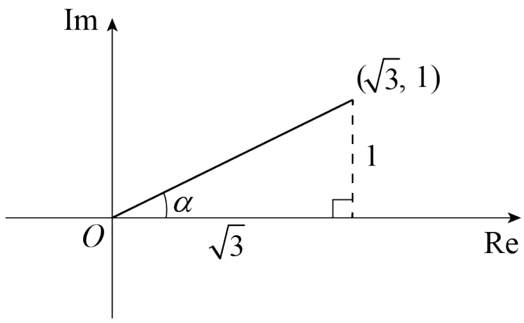


$|z| = \sqrt{(12^2 + 5^2)} = \sqrt{169} = 13$

$\tan\alpha = \frac{5}{12} \quad \alpha = 0.39$ rad.

$\arg z = 0.39$ radians (2 d.p.)

b $z = \sqrt{3} + \text{i}$

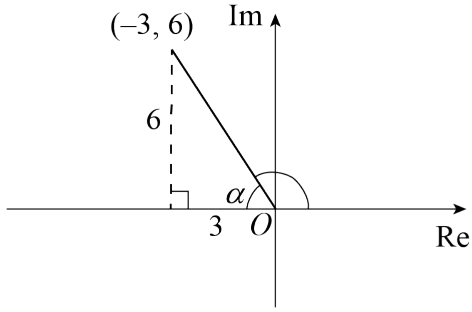


$|z| = \sqrt{\left((\sqrt{3})^2 + 1^2\right)} = \sqrt{4} = 2$

$\tan\alpha = \frac{1}{\sqrt{3}} \quad \alpha = \frac{\pi}{6}$

$\arg z = \frac{\pi}{6}$

c $z = -3 + 6\text{i}$

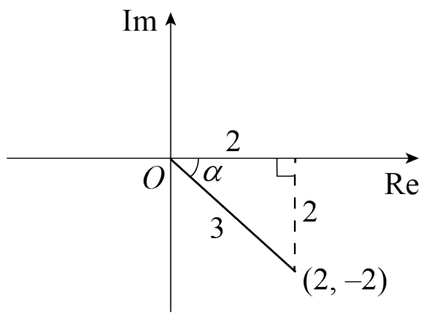


$|z| = \sqrt{\left((-3)^2 + 6^2\right)} = \sqrt{45} = 3\sqrt{5}$

$\tan\alpha = \frac{6}{3} \quad \alpha = 1.107$

$\arg z = \pi - \alpha = 2.03$ radians (2 d.p.)

d $z = 2 - 2\text{i}$

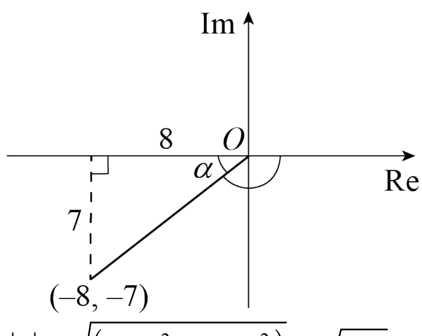


$|z| = \sqrt{\left(2^2 + (-2)^2\right)} = \sqrt{8} = 2\sqrt{2}$

$\tan\alpha = \frac{2}{2} \qquad \alpha = \frac{\pi}{4}$

$\arg z = -\alpha = -\frac{\pi}{4}$

e $z = -8 - 7\text{i}$

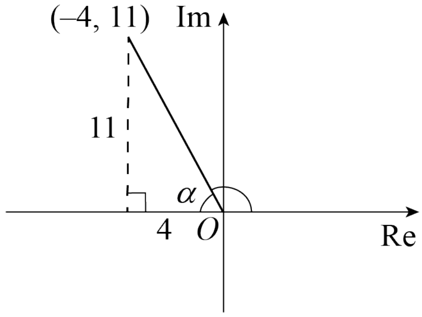


$|z| = \sqrt{\left((-8)^2 + (-7)^2\right)} = \sqrt{113}$

$\tan\alpha = \frac{7}{8} \quad \alpha = 0.7188$

$\arg z = -(\pi - \alpha) = -2.42$ radians (2 d.p.)

1 f $z = -4 + 11\text{i}$

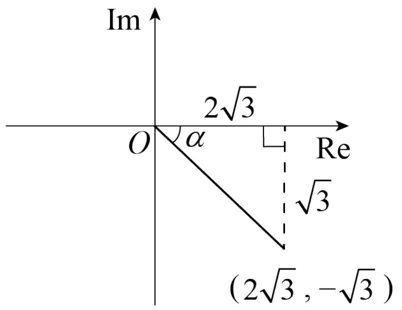


$|z| = \sqrt{((-4)^2 + 11^2)} = \sqrt{137}$

$\tan\ \alpha = \frac{11}{4} \quad \alpha = 1.222$

$\arg\ z = \pi - \alpha = 1.92$ radians (2 d.p.)

g $z = 2\sqrt{3} - \text{i}\sqrt{3}$

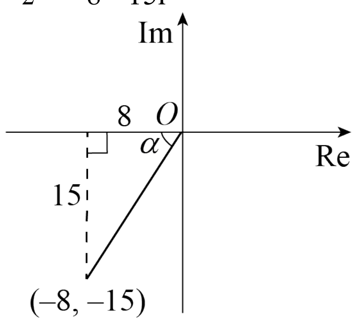


$|z| = \sqrt{((2\sqrt{3})^2 + (-\sqrt{3})^2)} = \sqrt{15}$

$\tan\ \alpha = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \quad \alpha = 0.4636$

$\arg\ z = -0.46$ radians (2 d.p.)

h $z = -8 - 15\text{i}$



$|z| = \sqrt{((-8)^2 + (-15)^2)} = \sqrt{289} = 17$

$\tan\ \alpha = \frac{15}{8} \quad \alpha = 1.0808$

$\arg\ z = -(\pi - \alpha) = -2.06$ radians (2 d.p.)

2 a i $|2 + 2\text{i}| = \sqrt{(2)^2 + (2)^2}$
$= \sqrt{8}$
$= 2\sqrt{2}$

ii $\tan\alpha = \frac{2}{2}$, so $\alpha = \frac{\pi}{4}$

$\arg z = \frac{\pi}{4}$

b i $|5 + 5\text{i}| = \sqrt{(5)^2 + (5)^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$

ii $\tan\alpha = \frac{5}{5}$, so $\alpha = \frac{\pi}{4}$

$\arg z = \frac{\pi}{4}$

c i $|-6 + 6\text{i}| = \sqrt{(-6)^2 + (6)^2}$
$= \sqrt{72}$
$= 6\sqrt{2}$

ii $\tan\alpha = \frac{6}{6}$, so $\alpha = \frac{\pi}{4}$

Here z is in the second quadrant, so

$\arg z = (\pi - \alpha)$

$\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

d i $|-a - a\text{i}| = \sqrt{(-a)^2 + (-a)^2}$
$= \sqrt{2a^2}$
$= a\sqrt{2}$

ii $\tan\alpha = \frac{a}{a}$, so $\alpha = \frac{\pi}{4}$

Here z is in the third quadrant, so

$\arg z = -(\pi - \alpha)$

$\arg z = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$

3 $z_1 = 3 + 5\text{i}$

a $|z_1| = |3+5\text{i}|$
$= \sqrt{3^2 + 5^2}$
$= \sqrt{34}$

b $z_1^2 = (3+5\text{i})^2$
$= 9 + 30\text{i} + 25\text{i}^2$
$= 9 + 30\text{i} - 25$
$= -16 + 30\text{i}$

c $|z_1^2| = |-16+30\text{i}|$
$= \sqrt{(-16)^2 + 30^2}$
$= 34$
$= (\sqrt{34})^2$
$= |z_1|^2$
Therefore, $|z_1^2| = |z_1|^2$ as required.

4 $z_1 = \dfrac{26}{3+2\text{i}}$

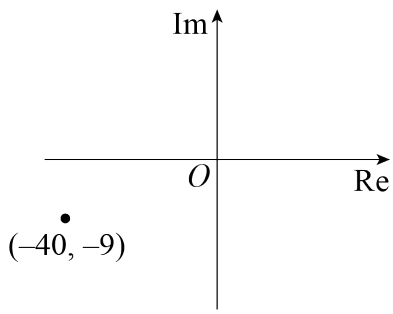
a $z_1 = \dfrac{26}{3+2\text{i}} \times \dfrac{3-2\text{i}}{3-2\text{i}}$
$= \dfrac{26(3-2\text{i})}{9-4\text{i}^2}$
$= \dfrac{26(3-2\text{i})}{13}$
$= 2(3-2\text{i})$
$= 6 - 4\text{i}$

b $|z_1| = |6-4\text{i}|$
$= \sqrt{6^2 + (-4)^2}$
$= \sqrt{52}$
$= 2\sqrt{13}$

c $|z_1 z_2| = |z_1||z_2| = 26\sqrt{13}$
$2\sqrt{13}|z_2| = 26\sqrt{13}$
$|z_2| = 13$

4 d $z_2 = 5 + p\text{i}$
$|z_2| = |5 + p\text{i}|$
$= \sqrt{5^2 + p^2}$
Since $|z_2| = 13$
$\sqrt{5^2 + p^2} = 13$
$5^2 + p^2 = 169$
$p^2 = 144$
$p = \pm 12$

5 a



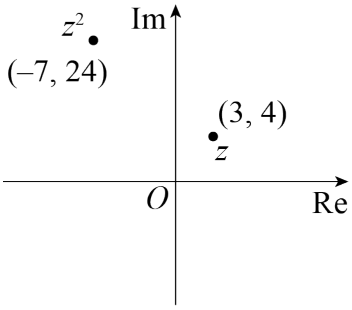
b $\tan\alpha = \dfrac{9}{40}$
$\alpha = \arctan\left(\dfrac{9}{40}\right) = 0.2213...$
Here z is in the third quadrant,
so $\arg z = -(\pi - \alpha)$
$\arg z = -(\pi - 0.2213...) = -2.92$

6 a $z = 3 + 4\text{i}$
$z^2 = (3+4\text{i})^2 = 9 + 12\text{i} + 12\text{i} + 16\text{i}^2$
$= -7 + 24\text{i}$

b $|z^2| = \sqrt{(-7)^2 + (24)^2} = \sqrt{625} = 25$

c $\tan\alpha = \dfrac{24}{7}$
$\alpha = \arctan\left(\dfrac{24}{7}\right) = 1.2870...$
z^2 is in the second quadrant, so
$\arg z^2 = \pi - \alpha$
$\arg z^2 = \pi - 1.2870... = 1.85$

6 d



7 a $\dfrac{z_1}{z_2} = \dfrac{4+6\text{i}}{1+\text{i}}$

Multiply by the complex conjugate

$$\frac{4+6\text{i}}{1+\text{i}} = \frac{(4+6\text{i})}{(1+\text{i})} \times \frac{(1-\text{i})}{(1-\text{i})}$$

$$= \frac{4-4\text{i}+6\text{i}-6\text{i}^2}{1-\text{i}+\text{i}-\text{i}^2}$$

$$= \frac{10}{2} + \frac{2}{2}\text{i}$$

$$= 5+\text{i}$$

b $\left|\dfrac{z_1}{z_2}\right| = \sqrt{(5)^2+(1)^2}$

$$= \sqrt{26}$$

c $\tan\alpha = \dfrac{1}{5}$

$$\alpha = \arctan\frac{1}{5} = 0.20$$

As z is in the first quadrant

$\arg z = \alpha = 0.20$ radians (2 d.p.)

8 a $\dfrac{z_1}{z_2} = 1-\text{i} \Rightarrow z_2 = \dfrac{z_1}{1-\text{i}}$

$$z_2 = \frac{3+2p\text{i}}{1-\text{i}}$$

Multiply by the complex conjugate

$$z_2 = \frac{(3+2p\text{i})}{(1-\text{i})} \times \frac{(1+\text{i})}{(1+\text{i})}$$

$$= \frac{3+3\text{i}+2p\text{i}+2p\text{i}^2}{1+\text{i}-\text{i}-\text{i}^2}$$

$$= \left(\frac{3-2p}{2}\right) + \left(\frac{3+2p}{2}\right)\text{i}$$

8 b $\arg z_2 = \arctan\left(\dfrac{\frac{3+2p}{2}}{\frac{3-2p}{2}}\right)$

$$= \arctan\left(\frac{3+2p}{3-2p}\right)$$

Since we are told that $\arg z_2 = \arctan 5$ it follows that:

$$\frac{3+2p}{3-2p} = 5$$

$$3+2p = 15-10p$$

$$-12 = -12p$$

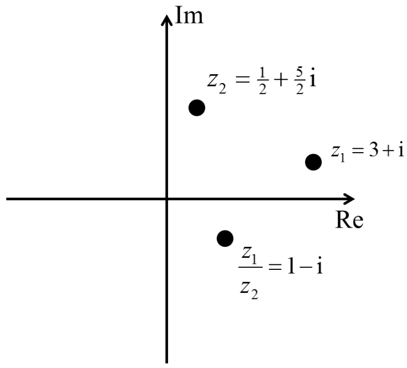
$$p = 1$$

c $z_2 = \dfrac{3-2p}{2} + \dfrac{3+2p}{2}\text{i}$

When $p = 1$, $z_2 = \dfrac{1}{2} + \dfrac{5}{2}\text{i}$

$$|z_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{26}{4}} = \frac{\sqrt{26}}{2}$$

d



9 a $z = \dfrac{26}{2-3\text{i}}$

Multiply by the complex conjugate

$$z = \frac{26}{(2-3\text{i})} \times \frac{(2+3\text{i})}{(2+3\text{i})}$$

$$= \frac{52+78\text{i}}{4+6\text{i}-6\text{i}-9\text{i}^2}$$

$$= \frac{52}{13} + \frac{78}{13}\text{i}$$

$$= 4+6\text{i}$$

b $z^2 = (4+6\text{i})^2 = 16+24\text{i}+24\text{i}+36\text{i}^2$

So $z^2 = -20+48\text{i}$

c $|z| = \sqrt{(4)^2+(6)^2} = \sqrt{52} = 2\sqrt{13}$

9 d $\tan\alpha = \frac{48}{20}$

$\alpha = \tan^{-1}\left(\frac{48}{20}\right) = 1.1760...$

Here z^2 is in the second quadrant

so $\arg\left(z^2\right) = \pi - \alpha$

$\arg\left(z^2\right) = \pi - 1.1760...$

$= 1.97$ radians (2 d.p.)

10 a $z_1 + z_2 = (4+2\mathrm{i}) + (2+4\mathrm{i}) = 6 + 6\mathrm{i}$

$|z_1 + z_2| = \sqrt{(6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$

b $w = \frac{z_1 z_3}{z_2} = \frac{(4+2\mathrm{i})(a+b\mathrm{i})}{2+4\mathrm{i}}$

$z_1 z_3 = (4+2\mathrm{i})(a+b\mathrm{i})$

$= 4a + 4b\mathrm{i} + 2a\mathrm{i} + 2b\mathrm{i}^2$

$= (4a-2b) + (4b+2a)\mathrm{i}$

$w = \frac{(4a-2b)+(4b+2a)\mathrm{i}}{2+4\mathrm{i}}$

Multiply by the complex conjugate

$w = \frac{((4a-2b)+(4b+2a)\mathrm{i})}{(2+4\mathrm{i})} \times \frac{(2-4\mathrm{i})}{(2-4\mathrm{i})}$

$= \frac{8a - 4b - (16a-8b)\mathrm{i} + (8b+4a)\mathrm{i} - (16b+8a)\mathrm{i}^2}{4 - 8\mathrm{i} + 8\mathrm{i} - 16\mathrm{i}^2}$

$= \frac{(16a+12b)+(-12a+16b)\mathrm{i}}{20}$

$= \frac{16a+12b}{20} + \frac{-12a+16b}{20}\mathrm{i}$

$= \frac{4a+3b}{5} + \frac{-3a+4b}{5}\mathrm{i}$

10 c $w = \frac{21}{5} - \frac{22}{5}\mathrm{i} = \frac{4a+3b}{5} + \frac{-3a+4b}{5}\mathrm{i}$

Equate real coefficients:

$4a + 3b = 21$ **(1)**

Equate imaginary coefficients:

$-3a + 4b = -22$ **(2)**

$3 \times$ **(1)**: $12a + 9b = 63$ **(3)**

$4 \times$ **(2)**: $-12a + 16b = -88$ **(4)**

(3) + **(4)** $\Rightarrow 25b = -25$

$b = -1$

Substituting $b = -1$ into (1): $4a - 3 = 21$

$a = 6$

d $\tan\alpha = \frac{\frac{22}{5}}{\frac{21}{5}} = \frac{22}{21}$

$\alpha = \arctan\left(\frac{22}{21}\right) = 0.8086...$

Here z is in the fourth quadrant, so $\arg z = -\alpha$

$\arg z = -0.81$ radians (2 d.p.)

11 a $|w| = \sqrt{(6)^2 + (3)^2} = \sqrt{45} = 3\sqrt{5}$

b $\tan\alpha = \frac{3}{6} = \frac{1}{2}$

$\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 0.4636...$

Here w is in the first quadrant

so $\arg z = \alpha$

$\arg z = 0.46$ radians (2 d.p.)

c $\arg(\lambda + 5\mathrm{i} + w)$

$= \arg(\lambda + 5\mathrm{i} + 6 + 3\mathrm{i})$

$= \arg((\lambda + 6) + 8\mathrm{i})$

$\tan\alpha = \frac{8}{\lambda + 6}$

As $\tan\frac{\pi}{4} = 1$, it follows that

$\frac{8}{\lambda+6} = 1$, so $\lambda = 2$

12 a $|z|=\sqrt{(-1)^2+\left(\sqrt{3}\right)^2}=\sqrt{4}=2$

b $z=-1-\mathrm{i}\sqrt{3}$, so $z^*=-1+\mathrm{i}\sqrt{3}$

$\frac{z}{z^*}=\frac{-1-\mathrm{i}\sqrt{3}}{-1+\mathrm{i}\sqrt{3}}$

Multiply by the complex conjugate

$\frac{z}{z^*}=\frac{\left(-1-\mathrm{i}\sqrt{3}\right)}{\left(-1+\mathrm{i}\sqrt{3}\right)}\times\frac{\left(-1-\mathrm{i}\sqrt{3}\right)}{\left(-1-\mathrm{i}\sqrt{3}\right)}$

$=\frac{1+\mathrm{i}\sqrt{3}+\mathrm{i}\sqrt{3}+\mathrm{i}^2\left(\sqrt{3}\right)^2}{1+\mathrm{i}\sqrt{3}-\mathrm{i}\sqrt{3}-\mathrm{i}^2\left(\sqrt{3}\right)^2}$

$=\frac{-2+2\mathrm{i}\sqrt{3}}{4}$

$=-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$

$\left|\frac{z}{z^*}\right|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{1}=1$

12 c For $\arg z$, $\tan\alpha=\frac{\sqrt{3}}{1}$

$\alpha=\arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$

Here z is in the third quadrant, so

$\arg z=-(\pi-\alpha)$

$\arg z=-\left(\pi-\frac{\pi}{3}\right)=-\frac{2\pi}{3}$

For $\arg z^*$, $\tan\alpha=\frac{\sqrt{3}}{1}$

$\alpha=\arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$

Here z^* is in the second quadrant, so $\arg z^*=\pi-\alpha$

$\arg z^*=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

For $\arg\left(\frac{z}{z^*}\right)$, $\tan\alpha=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\frac{2\sqrt{3}}{2}=\sqrt{3}$

$\alpha=\arctan\left(\sqrt{3}\right)=\frac{\pi}{3}$

Here $\frac{z}{z^*}$ is in the second quadrant, so $\arg\left(\frac{z}{z^*}\right)=\pi-\alpha$

$\arg\left(\frac{z}{z^*}\right)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

13 $w+z=(k+\mathrm{i})+(-4+5k\mathrm{i})$

$=(k-4)+(1+5k)\mathrm{i}$

$\arg(w+z)=\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}=-\sqrt{3}$

$\frac{1+5k}{k-4}=\frac{-\sqrt{3}}{1}$

$1+5k=-k\sqrt{3}+4\sqrt{3}$

So $5k+k\sqrt{3}=4\sqrt{3}-1$

$k\left(5+\sqrt{3}\right)=4\sqrt{3}-1$

$k=\frac{4\sqrt{3}-1}{5+\sqrt{3}}$

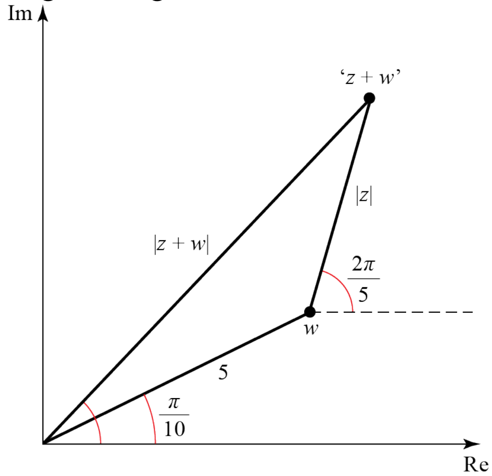
Rationalise the denominator by multiplying by the conjugate

$k=\frac{\left(4\sqrt{3}-1\right)}{\left(5+\sqrt{3}\right)}\times\frac{\left(5-\sqrt{3}\right)}{\left(5-\sqrt{3}\right)}$

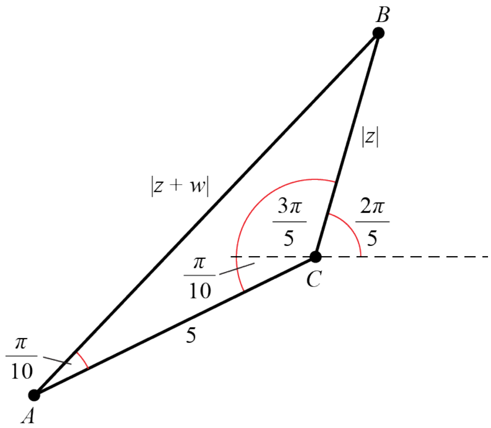
$=\frac{20\sqrt{3}-12-5+\sqrt{3}}{25-5\sqrt{3}+5\sqrt{3}-3}$

$=\frac{21\sqrt{3}-17}{22}$

14 Represent the given information on an Argand diagram:



Using this information, consider the triangle ΔABC shown below:



$\angle A=\frac{\pi}{5}-\frac{\pi}{10}=\frac{\pi}{10}$

$\angle C=\frac{3\pi}{5}+\frac{\pi}{10}=\frac{7\pi}{10}$

The angles in a triangle sum to π, so

$\angle B=\pi-\angle C-\angle A$

$=\pi-\frac{7\pi}{10}-\frac{\pi}{10}$

$=\frac{\pi}{5}$

Using the sine rule:

$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

$\frac{\sin\frac{\pi}{10}}{|z|}=\frac{\sin\frac{\pi}{5}}{5}=\frac{\sin\frac{7\pi}{10}}{|z+w|}$

$5\sin\frac{\pi}{10}=|z|\sin\frac{\pi}{5}$

$|z|=\frac{5\sin\frac{\pi}{10}}{\sin\frac{\pi}{5}}\approx 2.63$