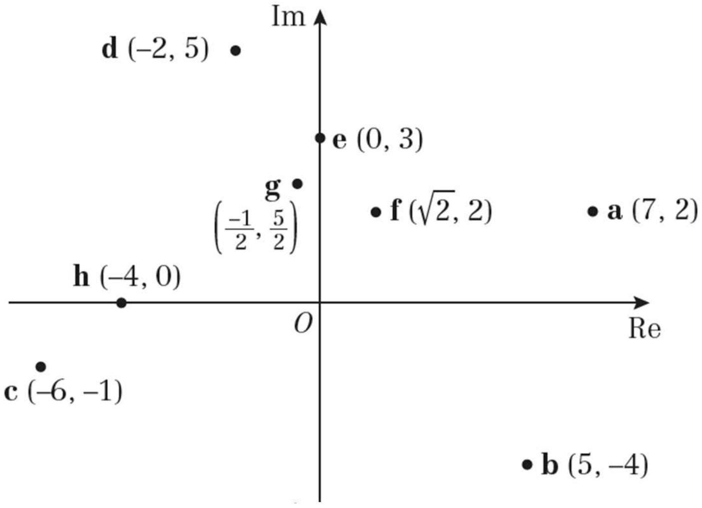
Exercise 1E

1



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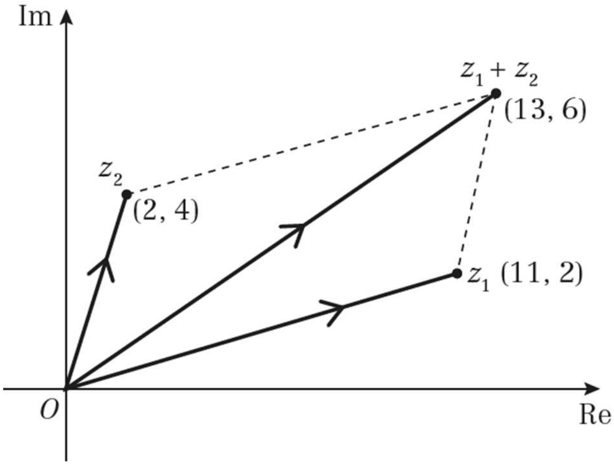

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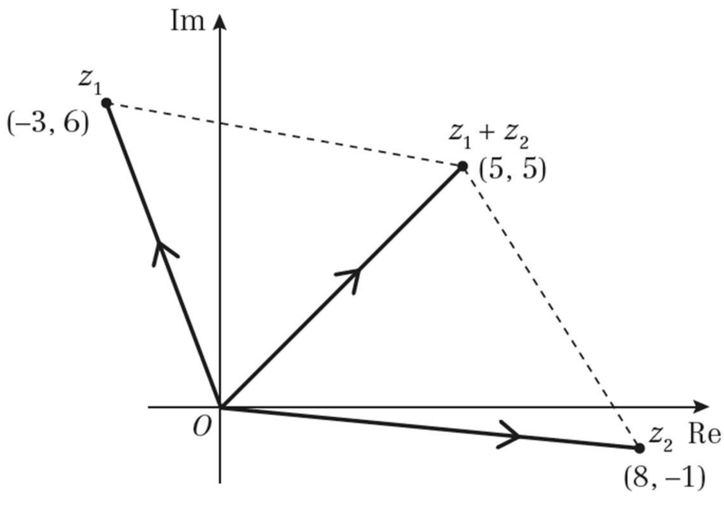

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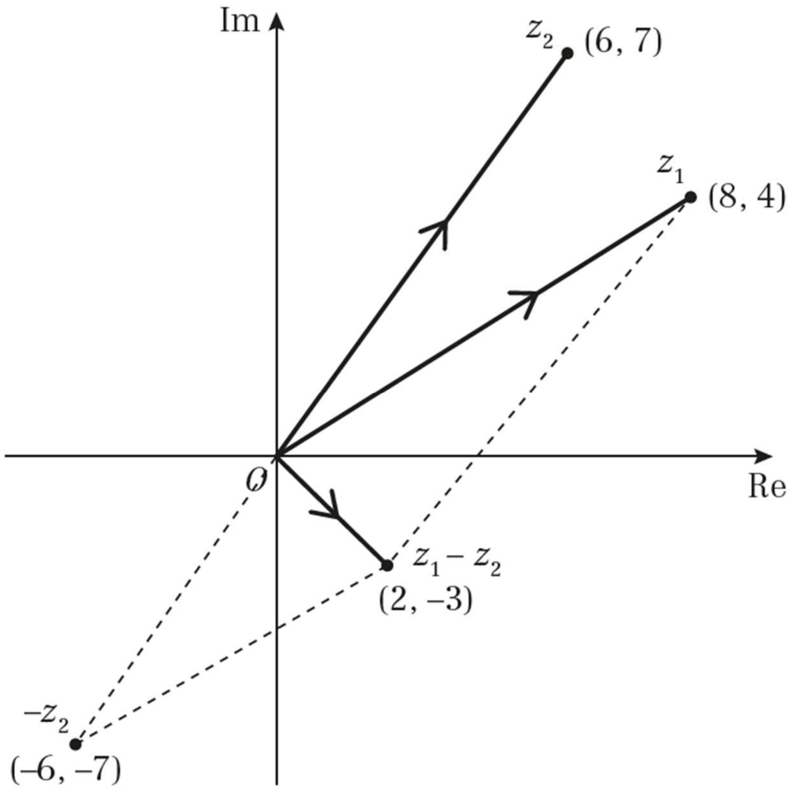

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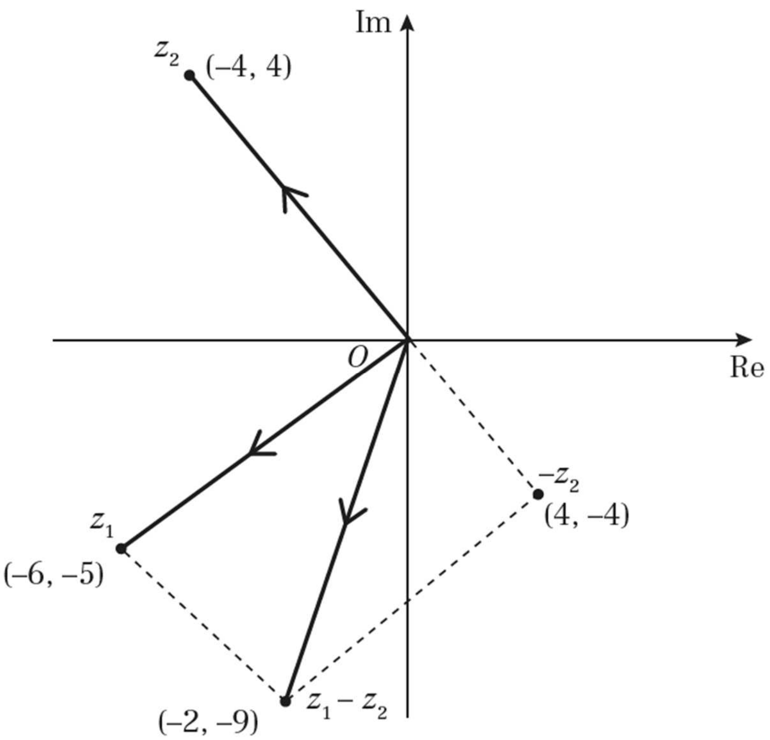


6 **a** $z_3 = z_1 + z_2$

$-3+2\text{i} = (7-5\text{i}) + (a+b\text{i})$

$= (7+a) + (-5+b)\text{i}$

Equate real coefficients: $-3 = 7 + a$

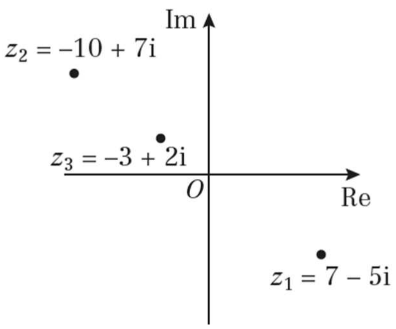
$a = -10$

Equate imaginary coefficients: $2 = -5 + b$

$b = 7$

Hence $z_2 = -10 + 7\text{i}$

b



7 **a** $z_3 = z_1 + z_2$

$-8+5\text{i} = (p+q\text{i}) + (9-5\text{i})$

Equate real coefficients: $-8 = p + 9$

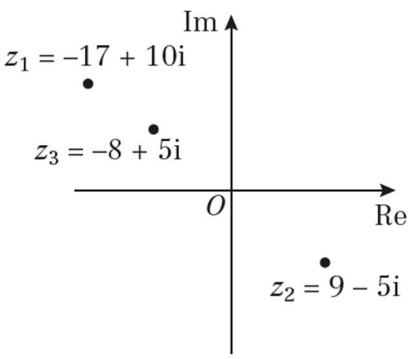
$p = -17$

Equate imaginary coefficients: $5 = -5 + q$

$q = 10$

$z_1 = -17 + 10\text{i}$

b



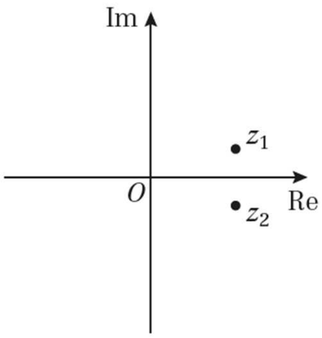
8 a $z^2 - 6z + 10 = 0$

Solve by completing the square:

$$(z-3)^2 - 9 + 10 = 0$$
$$(z-3)^2 = -1$$
$$z - 3 = \pm \mathrm{i}$$
$$z = 3 \pm \mathrm{i}$$

So $z_1 = 3 + \mathrm{i}$ and $z_2 = 3 - \mathrm{i}$.

b



9 a $\mathrm{f}(z) = 2z^3 - 19z^2 + 64z - 60$

$$\begin{aligned}\mathrm{f}\left(\tfrac{3}{2}\right) &= 2\left(\tfrac{3}{2}\right)^3 - 19\left(\tfrac{3}{2}\right)^2 + 64\left(\tfrac{3}{2}\right) - 60 \\ &= 2\left(\tfrac{27}{8}\right) - 19\left(\tfrac{9}{4}\right) + 64\left(\tfrac{3}{2}\right) - 60 \\ &= \tfrac{54}{8} - \tfrac{171}{4} + \tfrac{192}{2} - 60 \\ &= \tfrac{27}{4} - \tfrac{171}{4} + \tfrac{384}{4} - \tfrac{240}{4} \\ &= 0\end{aligned}$$

b If $\mathrm{f}\left(\tfrac{3}{2}\right) = 0$, then $2z - 3$ is a factor of $\mathrm{f}(z)$.

Use division to find the other factors:

$$\begin{array}{r} z^2 - 8z + 20 \\ 2z-3\,\overline{)\,2z^3 - 19z^2 + 64z - 60} \\ 2z^3 - 3z^2 \qquad\qquad\quad \\ -16z^2 + 64z \qquad \\ -16z^2 + 24z \qquad \\ 40z - 60 \\ 40z - 60 \\ 0 \end{array}$$

So $2z^3 - 19z^2 + 64z - 60 = (2z-3)(z^2 - 8z + 20)$

Either $2z - 3 = 0 \Rightarrow z = \tfrac{3}{2}$

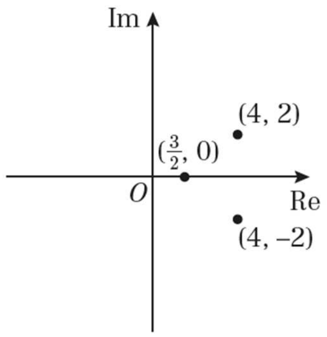
or $z^2 - 8z + 20 = 0$

9 b Solve by completing the square:

$$(z-4)^2 - 16 + 20 = 0$$
$$(z-4)^2 = -4$$
$$z - 4 = \pm 2\text{i}$$
$$z = 4 \pm 2\text{i}$$

So the roots of $\text{f}(z) = 0$ are $\frac{3}{2}$, $4+2\text{i}$ and $4-2\text{i}$

c



Challenge

a

$$z^6 = 1$$
$$z^6 - 1 = 0$$
$$(z^3-1)(z^3+1) = 0 \quad (*)$$

Let $\text{f}(z) = z^3 - 1$.
Since $\text{f}(1) = 0$, then $z-1$ is a factor of $\text{f}(z)$.
Hence $\text{f}(z) = (z-1)(z^2+bz+c)$
Equate coefficients of z^2: $0 = b - 1$
$b = 1$
Equate constants: $-1 = -c$
$c = 1$
So $\text{f}(z) = (z-1)(z^2+z+1)$

Let $\text{g}(z) = z^3 + 1$.
Since $\text{g}(-1) = 0$, then $z+1$ is a factor of $\text{g}(z)$
Hence $\text{g}(z) = (z+1)(z^2+pz+q)$
Equate coefficients of z^2: $0 = 1 + p$
$p = -1$
Equate constants: $q = 1$
So $\text{g}(z) = (z+1)(z^2-z+1)$

Challenge

a By (*), $0 = \mathrm{f}(z)\mathrm{g}(z)$

$$0 = (z-1)(z^2+z+1)(z+1)(z^2-z+1)$$

Either $z-1=0 \Rightarrow z=1$

or

$$z^2+z+1=0$$

$$\left(z+\frac{1}{2}\right)^2-\frac{1}{4}+1=0$$

$$\left(z+\frac{1}{2}\right)^2=-\frac{3}{4}$$

$$z+\frac{1}{2}=\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

$$z=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

or $z+1=0 \Rightarrow z=-1$

or

$$z^2-z+1=0$$

$$\left(z-\frac{1}{2}\right)^2-\frac{1}{4}+1=0$$

$$\left(z-\frac{1}{2}\right)^2=-\frac{3}{4}$$

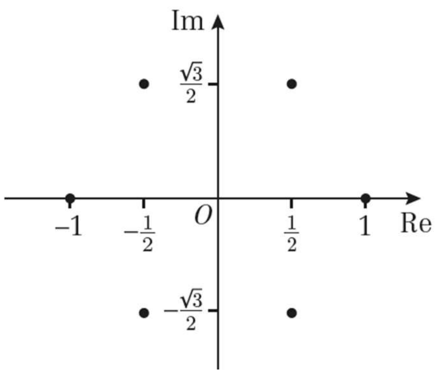
$$z-\frac{1}{2}=\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

$$z=\frac{1}{2}\pm\frac{\sqrt{3}}{2}\mathrm{i}$$

So the roots of $z^6=1$

are $-1,\ 1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$ and $\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$

b



Challenge

c $(0,1)$ and $(0,-1)$ are on the unit circle.

Use Pythagoras' Theorem to check $\pm\frac{1}{2}\pm\frac{\sqrt{3}}{2}\text{i}$ also lie on a circle with centre $(0,0)$ and radius 1.

$$\left(\pm\frac{1}{2}\right)^2+\left(\pm\frac{\sqrt{3}}{2}\right)^2=\frac{1}{4}+\frac{3}{4}=1$$

So all points lie on a circle with centre $(0,0)$ and radius 1.