Exercise 1D

1 a $z^* = 8 - 2\text{i}$

b $z^* = 6 + 5\text{i}$

c $z^* = \frac{2}{3} + \frac{1}{2}\text{i}$

d $z^* = \sqrt{5} - \text{i}\sqrt{10}$

2 a
$$\begin{aligned} z + z^* &= (6 - 3\text{i}) + (6 + 3\text{i}) \\ &= 12 \end{aligned}$$

$$\begin{aligned} zz^* &= (6 - 3\text{i})(6 + 3\text{i}) \\ &= 6(6 + 3\text{i}) - 3\text{i}(6 + 3\text{i}) \\ &= 36 + 18\text{i} - 18\text{i} - 9\text{i}^2 \\ &= (36 + 9) + \text{i}(18 - 18) \\ &= 45 \end{aligned}$$

b
$$\begin{aligned} z + z^* &= (10 + 5\text{i}) + (10 - 5\text{i}) \\ &= 20 \end{aligned}$$

$$\begin{aligned} zz^* &= (10 + 5\text{i})(10 - 5\text{i}) \\ &= 10(10 - 5\text{i}) + 5\text{i}(10 - 5\text{i}) \\ &= 100 - 50\text{i} + 50\text{i} - 25\text{i}^2 \\ &(100 + 25) + \text{i}(-50 + 50) \\ &= 125 \end{aligned}$$

c
$$\begin{aligned} z + z^* &= \left(\tfrac{3}{4} + \tfrac{1}{4}\text{i}\right) + \left(\tfrac{3}{4} - \tfrac{1}{4}\text{i}\right) \\ &= \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} zz^* &= \left(\tfrac{3}{4} + \tfrac{1}{4}\text{i}\right)\left(\tfrac{3}{4} - \tfrac{1}{4}\text{i}\right) \\ &= \tfrac{3}{4}\left(\tfrac{3}{4} - \tfrac{1}{4}\text{i}\right) + \tfrac{1}{4}\text{i}\left(\tfrac{3}{4} - \tfrac{1}{4}\text{i}\right) \\ &= \tfrac{9}{16} - \tfrac{3}{16}\text{i} + \tfrac{3}{16}\text{i} - \tfrac{1}{16}\text{i}^2 \\ &= \tfrac{10}{16} = \tfrac{5}{8} \end{aligned}$$

2 d
$$\begin{aligned} z + z^* &= \left(\sqrt{5} - 3\text{i}\sqrt{5}\right) + \left(\sqrt{5} + 3\text{i}\sqrt{5}\right) \\ &= 2\sqrt{5} \end{aligned}$$

$$\begin{aligned} zz^* &= \left(\sqrt{5} - 3\text{i}\sqrt{5}\right)\left(\sqrt{5} + 3\text{i}\sqrt{5}\right) \\ &= \sqrt{5}\left(\sqrt{5} + 3\text{i}\sqrt{5}\right) - 3\text{i}\sqrt{5}\left(\sqrt{5} + 3\text{i}\sqrt{5}\right) \\ &= 5 + 15\text{i} - 15\text{i} - 45\text{i}^2 \\ &= (5 + 45) + \text{i}(15 - 15) \\ &= 50 \end{aligned}$$

3 a
$$\frac{3 - 5\text{i}}{1 + 3\text{i}} = \frac{(3 - 5\text{i})(1 - 3\text{i})}{(1 + 3\text{i})(1 - 3\text{i})}$$

$$\begin{aligned} (3 - 5\text{i})(1 - 3\text{i}) &= 3(1 - 3\text{i}) - 5\text{i}(1 - 3\text{i}) \\ &= 3 - 9\text{i} - 5\text{i} + 15\text{i}^2 \\ &= -12 - 14\text{i} \end{aligned}$$

$$\begin{aligned} (1 + 3\text{i})(1 - 3\text{i}) &= 1(1 - 3\text{i}) + 3\text{i}(1 - 3\text{i}) \\ &= 1 - 3\text{i} + 3\text{i} - 9\text{i}^2 \\ &= 10 \end{aligned}$$

$$\frac{3 - 5\text{i}}{1 + 3\text{i}} = \frac{-12 - 14\text{i}}{10} = -\frac{6}{5} - \frac{7}{5}\text{i}$$

b
$$\frac{3 + 5\text{i}}{6 - 8\text{i}} = \frac{(3 + 5\text{i})(6 + 8\text{i})}{(6 - 8\text{i})(6 + 8\text{i})}$$

$$\begin{aligned} (3 + 5\text{i})(6 + 8\text{i}) &= 3(6 + 8\text{i}) + 5\text{i}(6 + 8\text{i}) \\ &= 18 + 24\text{i} + 30\text{i} + 40\text{i}^2 \\ &= -22 + 54\text{i} \end{aligned}$$

$$\begin{aligned} (6 - 8\text{i})(6 + 8\text{i}) &= 6(6 + 8\text{i}) - 8\text{i}(6 + 8\text{i}) \\ &= 36 + 48\text{i} - 48\text{i} - 64\text{i}^2 \\ &= 100 \end{aligned}$$

$$\frac{3 + 5\text{i}}{6 - 8\text{i}} = \frac{-22 + 54\text{i}}{100} = -\frac{11}{50} + \frac{27}{50}\text{i}$$

3 c
$$\frac{28-3\mathrm{i}}{1-\mathrm{i}}=\frac{(28-3\mathrm{i})(1+\mathrm{i})}{(1-\mathrm{i})(1+\mathrm{i})}$$
$$\begin{aligned}(28-3\mathrm{i})(1+\mathrm{i})&=28(1+\mathrm{i})-3\mathrm{i}(1+\mathrm{i})\\&=28+28\mathrm{i}-3\mathrm{i}-3\mathrm{i}^2\\&=31+25\mathrm{i}\end{aligned}$$
$$\begin{aligned}(1-\mathrm{i})(1+\mathrm{i})&=1(1+\mathrm{i})-\mathrm{i}(1+\mathrm{i})\\&=1+\mathrm{i}-\mathrm{i}-\mathrm{i}^2\\&=2\end{aligned}$$
$$\frac{28-3\mathrm{i}}{1-\mathrm{i}}=\frac{31+25\mathrm{i}}{2}=\frac{31}{2}+\frac{25}{2}\mathrm{i}$$

d
$$\frac{2+\mathrm{i}}{1+4\mathrm{i}}=\frac{(2+\mathrm{i})(1-4\mathrm{i})}{(1+4\mathrm{i})(1-4\mathrm{i})}$$
$$\begin{aligned}(2+\mathrm{i})(1-4\mathrm{i})&=2(1-4\mathrm{i})+\mathrm{i}(1-4\mathrm{i})\\&=2-8\mathrm{i}+\mathrm{i}-4\mathrm{i}^2\\&=6-7\mathrm{i}\end{aligned}$$
$$\begin{aligned}(1+4\mathrm{i})(1-4\mathrm{i})&=1(1-4\mathrm{i})+4\mathrm{i}(1-4\mathrm{i})\\&=1-4\mathrm{i}+4\mathrm{i}-16\mathrm{i}^2\\&=17\end{aligned}$$
$$\frac{2+\mathrm{i}}{1+4\mathrm{i}}=\frac{6-7\mathrm{i}}{17}=\frac{6}{17}-\frac{7}{17}\mathrm{i}$$

4
$$\begin{aligned}\frac{(3-4\mathrm{i})^2}{1+\mathrm{i}}&=\frac{\left(9-24\mathrm{i}+16\mathrm{i}^2\right)}{(1+\mathrm{i})}\times\frac{(1-\mathrm{i})}{(1-\mathrm{i})}\\&=\frac{9-24\mathrm{i}+16\mathrm{i}^2-9\mathrm{i}+24\mathrm{i}^2-16\mathrm{i}^3}{1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2}\\&=\frac{9-24\mathrm{i}-16-9\mathrm{i}-24+16\mathrm{i}}{1-(-1)}\\&=\frac{-31-17\mathrm{i}}{2}\\&=-\frac{31}{2}-\frac{17}{2}i\end{aligned}$$

5 a
$$\begin{aligned}z_1z_2&=(1+\mathrm{i})(2+\mathrm{i})\\&=1(2+\mathrm{i})+\mathrm{i}(2+\mathrm{i})\\&=2+\mathrm{i}+2\mathrm{i}+\mathrm{i}^2\\&=1+3\mathrm{i}\end{aligned}$$
$$\frac{z_1z_2}{z_3}=\frac{1+3\mathrm{i}}{3+\mathrm{i}}=\frac{(1+3\mathrm{i})(3-\mathrm{i})}{(3+\mathrm{i})(3-\mathrm{i})}$$
$$\begin{aligned}(1+3\mathrm{i})(3-\mathrm{i})&=1(3-\mathrm{i})+3\mathrm{i}(3-\mathrm{i})\\&=3-\mathrm{i}+9\mathrm{i}-3\mathrm{i}^2\\&=6+8\mathrm{i}\end{aligned}$$
$$\begin{aligned}(3+\mathrm{i})(3-\mathrm{i})&=3(3-\mathrm{i})+\mathrm{i}(3-\mathrm{i})\\&=9-3\mathrm{i}+3\mathrm{i}-\mathrm{i}^2\\&=10\end{aligned}$$
$$\frac{z_1z_2}{z_3}=\frac{6+8\mathrm{i}}{10}=\frac{3}{5}+\frac{4}{5}\mathrm{i}$$

b
$$\begin{aligned}(z_2)^2&=(2+\mathrm{i})(2+\mathrm{i})\\&=2(2+\mathrm{i})+\mathrm{i}(2+\mathrm{i})\\&=4+2\mathrm{i}+2\mathrm{i}+\mathrm{i}^2\\&=3+4\mathrm{i}\end{aligned}$$
$$\frac{(z_2)^2}{z_1}=\frac{3+4\mathrm{i}}{1+\mathrm{i}}=\frac{(3+4\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$$
$$\begin{aligned}(3+4\mathrm{i})(1-\mathrm{i})&=3(1-\mathrm{i})+4\mathrm{i}(1-\mathrm{i})\\&=3-3\mathrm{i}+4\mathrm{i}-4\mathrm{i}^2\\&=7+\mathrm{i}\end{aligned}$$
$$\begin{aligned}(1+\mathrm{i})(1-\mathrm{i})&=1(1-\mathrm{i})+\mathrm{i}(1-\mathrm{i})\\&=1-\mathrm{i}+\mathrm{i}-\mathrm{i}^2\\&=2\end{aligned}$$
$$\frac{(z_2)^2}{z_1}=\frac{7+\mathrm{i}}{2}=\frac{7}{2}+\frac{1}{2}\mathrm{i}$$

5 c

$$2z_1 + 5z_3 = 2(1+\text{i}) + 5(3+\text{i})$$
$$= 2 + 2\text{i} + 15 + 5\text{i}$$
$$= 17 + 7\text{i}$$
$$\frac{2z_1 + 5z_3}{z_2} = \frac{17+7\text{i}}{2+\text{i}} = \frac{(17+7\text{i})(2-\text{i})}{(2+\text{i})(2-\text{i})}$$
$$(17+7\text{i})(2-\text{i}) = 17(2-\text{i}) + 7\text{i}(2-\text{i})$$
$$= 34 - 17\text{i} + 14\text{i} - 7\text{i}^2$$
$$= 41 - 3\text{i}$$
$$(2+\text{i})(2-\text{i}) = 2(2-\text{i}) + \text{i}(2-\text{i})$$
$$= 4 - 2\text{i} + 2\text{i} - \text{i}^2$$
$$= 5$$
$$\frac{2z_1 + 5z_3}{z_2} = \frac{41-3\text{i}}{5} = \frac{41}{5} - \frac{3}{5}\text{i}$$

6

$$\frac{5+2\text{i}}{z} = 2 - \text{i}$$
$$z = \frac{5+2\text{i}}{2-\text{i}} = \frac{(5+2\text{i})(2+\text{i})}{(2-\text{i})(2+\text{i})}$$
$$(5+2\text{i})(2+\text{i}) = 5(2+\text{i}) + 2\text{i}(2+\text{i})$$
$$= 10 + 5\text{i} + 4\text{i} + 2\text{i}^2$$
$$= 8 + 9\text{i}$$
$$(2-\text{i})(2+\text{i}) = 2(2+\text{i}) - \text{i}(2+\text{i})$$
$$= 4 + 2\text{i} - 2\text{i} - \text{i}^2$$
$$= 5$$
$$z = \frac{8+9\text{i}}{5} = \frac{8}{5} + \frac{9}{5}\text{i}$$

7

$$\frac{6+8\text{i}}{1+\text{i}} + \frac{6+8\text{i}}{1-\text{i}}$$
$$= \frac{(6+8\text{i})(1-\text{i}) + (6+8\text{i})(1+\text{i})}{(1+\text{i})(1-\text{i})}$$
$$= \frac{6(1-\text{i}) + 8\text{i}(1-\text{i}) + 6(1+\text{i}) + 8\text{i}(1+\text{i})}{1(1-\text{i}) + \text{i}(1-\text{i})}$$
$$= \frac{6 - 6\text{i} + 8\text{i} - 8\text{i}^2 + 6 + 6\text{i} + 8\text{i} + 8\text{i}^2}{1 - \text{i} + \text{i} - \text{i}^2}$$
$$= \frac{12+16\text{i}}{2} = 6 + 8\text{i}$$

8

$$\frac{4}{8-\text{i}\sqrt{2}} = \frac{4}{\left(8-\text{i}\sqrt{2}\right)} \times \frac{\left(8+\text{i}\sqrt{2}\right)}{\left(8+\text{i}\sqrt{2}\right)}$$
$$= \frac{32 + 4\text{i}\sqrt{2}}{64 + 8\text{i}\sqrt{2} - 8\text{i}\sqrt{2} - \text{i}^2\left(\sqrt{2}\right)^2}$$
$$= \frac{32 + 4\text{i}\sqrt{2}}{66}$$
$$= \frac{16}{33} + \frac{2\sqrt{2}}{33}\text{i}$$

9

$$\frac{1}{w} = \frac{1}{1-9\text{i}}$$
$$= \frac{1}{(1-9\text{i})} \times \frac{(1+9\text{i})}{(1+9\text{i})}$$
$$= \frac{1+9\text{i}}{1 + 9\text{i} - 9\text{i} - 81\text{i}^2}$$
$$= \frac{1+9\text{i}}{82}$$
$$= \tfrac{1}{82} + \tfrac{9}{82}\text{i}$$

10

$$\frac{z+4}{z-3} = \frac{4 - \text{i}\sqrt{2} + 4}{4 - \text{i}\sqrt{2} - 3}$$
$$= \frac{8 - \text{i}\sqrt{2}}{1 - \text{i}\sqrt{2}}$$
$$= \frac{\left(8-\text{i}\sqrt{2}\right)}{\left(1-\text{i}\sqrt{2}\right)} \times \frac{\left(1+\text{i}\sqrt{2}\right)}{\left(1+\text{i}\sqrt{2}\right)}$$
$$= \frac{8 + 8\text{i}\sqrt{2} - \text{i}\sqrt{2} - \text{i}^2\left(\sqrt{2}\right)^2}{1 + \text{i}\sqrt{2} - \text{i}\sqrt{2} - \text{i}^2\left(\sqrt{2}\right)^2}$$
$$= \frac{8 + 7\text{i}\sqrt{2} + 2}{1+2}$$
$$= \frac{10}{3} + \frac{7\sqrt{2}}{3}\text{i}$$

11 $(4+2\text{i})(z-2\text{i})=6-4\text{i}$

$$z-2\text{i}=\frac{6-4\text{i}}{4+2\text{i}}=\frac{(6-4\text{i})}{(4+2\text{i})}\times\frac{(4-2\text{i})}{(4-2\text{i})}$$

$$z-2\text{i}=\frac{24-12\text{i}-16\text{i}+8\text{i}^2}{16-8\text{i}+8\text{i}-(2\text{i})^2}$$

$$z-2\text{i}=\frac{16-28\text{i}}{20}$$

$$z-2\text{i}=\tfrac{4}{5}-\tfrac{7}{5}\text{i}$$

$$z=\tfrac{4}{5}+\tfrac{3}{5}\text{i}$$

Alternative Method:
Let $z=a+b\text{i}$
Substitute $z=a+b\text{i}$ into the equation $6-4\text{i}=(4+2\text{i})(z-2\text{i})$:

$$\begin{aligned}6-4\text{i}&=(4+2\text{i})(a+(b-2)\text{i})\\&=4a+4(b-2)\text{i}+2a\text{i}+2(b-2)\text{i}^2\\&=4a+4b\text{i}-8\text{i}+2a\text{i}-2(b-2)\\&=4a+4b\text{i}-8\text{i}+2a\text{i}-2b+4\\&=(4a-2b+4)+(2a+4b-8)\text{i}\end{aligned}$$

Equate real parts:
$6=4a-2b+4\Rightarrow 2a-b=1$ **(1)**
Equate imaginary parts:
$-4=2a+4b-8\Rightarrow a+2b=2$ **(2)**

$(2)+2\times(1)\Rightarrow 5a=4$

$a=\tfrac{4}{5}$

Substitute $a=\tfrac{4}{5}$ into (**2**)

$\tfrac{4}{5}+2b=2$

$b=\tfrac{3}{5}$

Hence $z=\tfrac{4}{5}+\tfrac{3}{5}\text{i}$

12 $$\frac{z_1}{z_2}=\frac{p-7\text{i}}{2+5\text{i}}$$

$$=\frac{(p-7\text{i})}{(2+5\text{i})}\times\frac{(2-5\text{i})}{(2-5\text{i})}$$

$$=\frac{2p-5p\text{i}-14\text{i}+35\text{i}^2}{4-10\text{i}+10\text{i}-25\text{i}^2}$$

$$=\frac{2p-35}{29}+\frac{-5p-14}{29}\text{i}$$

13 $z=\sqrt{5}+4\text{i}$

$z^*=\sqrt{5}-4\text{i}$

$$\frac{z}{z^*}=\frac{(\sqrt{5}+4\text{i})}{(\sqrt{5}-4\text{i})}\times\frac{(\sqrt{5}+4\text{i})}{(\sqrt{5}+4\text{i})}$$

$$=\frac{5+4\sqrt{5}\text{i}+4\sqrt{5}\text{i}+16\text{i}^2}{5+4\sqrt{5}\text{i}-4\sqrt{5}\text{i}-16\text{i}^2}$$

$$=-\frac{11}{21}+\frac{8\sqrt{5}}{21}\text{i}$$

14 a $$\frac{p+5\text{i}}{p-2\text{i}}=\frac{(p+5\text{i})}{(p-2\text{i})}\times\frac{(p+2\text{i})}{(p+2\text{i})}$$

$$=\frac{p^2+2p\text{i}+5p\text{i}+10\text{i}^2}{p^2+2p\text{i}-2p\text{i}-4\text{i}^2}$$

$$=\frac{p^2-10}{p^2+4}+\frac{7p}{p^2+4}\text{i} \quad \textbf{(1)}$$

The real part is equal to $\tfrac{1}{2}$, so

$$\frac{p^2-10}{p^2+4}=\frac{1}{2}$$

$2p^2-20=p^2+4$

$p^2=24$

$p=\pm2\sqrt{6}$

But since $p>0$, then $p=2\sqrt{6}$

b Substituting $p=2\sqrt{6}$ into (**1**):

$$z=\frac{(2\sqrt{6})^2-10}{(2\sqrt{6})^2+4}+\frac{7(2\sqrt{6})}{(2\sqrt{8})^2+4}\text{i}$$

$$=\frac{24-10}{24+4}+\frac{14\sqrt{6}}{24+4}\text{i}$$

$$=\frac{1}{2}+\frac{\sqrt{6}}{2}\text{i}$$