

## Exercise 1C

1 a

$$\begin{aligned}
 (5+i)(3+4i) &= 5(3+4i) + i(3+4i) \\
 &= 15 + 20i + 3i + 4i^2 \\
 &= 15 + 20i + 3i - 4 \\
 &= (15-4) + i(20+3) \\
 &= 11 + 23i
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (6+3i)(7+2i) &= 6(7+2i) + 3i(7+2i) \\
 &= 42 + 12i + 21i + 6i^2 \\
 &= 42 + 12i + 21i - 6 \\
 &= (42-6) + i(12+21) \\
 &= 36 + 33i
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (5-2i)(1+5i) &= 5(1+5i) - 2i(1+5i) \\
 &= 5 + 25i - 2i - 10i^2 \\
 &= 5 + 25i - 2i + 10 \\
 &= (5+10) + i(25-2) \\
 &= 15 + 23i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } (13-3i)(2-8i) &= 13(2-8i) - 3i(2-8i) \\
 &= 26 - 104i - 6i + 24i^2 \\
 &= 26 - 104i - 6i - 24 \\
 &= (26+24) + i(104-6) \\
 &= 2 - 110i
 \end{aligned}$$

$$\begin{aligned}
 \text{e } (-3-i)(4+7i) &= -3(4+7i) - i(4+7i) \\
 &= -12 - 21i - 4i - 7i^2 \\
 &= -12 - 21i - 4i + 7 \\
 &= (-12+7) + i(-21-4) \\
 &= -5 - 25i
 \end{aligned}$$

$$\begin{aligned}
 \text{f } (8+5i)(8+5i) &= 8(8+5i) + 5i(8+5i) \\
 &= 64 + 40i + 40i + 25i^2 \\
 &= 64 + 40i + 40i - 25 \\
 &= (64-25) + i(40+40) \\
 &= 39 + 80i
 \end{aligned}$$

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$$\begin{aligned}
 (2-9i)(2-9i) &= 2(2-9i) - 9i(2-9i) \\
 &= 4 - 18i - 18i + 81i^2 \\
 &= 4 - 18i - 18i - 81 \\
 &= (4-81) + i(-18-18) \\
 &= -77 - 36i
 \end{aligned}$$

$$\begin{aligned}
 \text{h } (1+i)(2+i)(3+i) &= (1+i)[2(3+i) + i(3+i)] \\
 &= (1+i)[6 + 2i + 3i + i^2] \\
 &= (1+i)[6 + 2i + 3i - 1] \\
 &= (1+i)(5+5i) \\
 &= 1(5+5i) + i(5+5i) \\
 &= 5 + 5i + 5i + 5i^2 \\
 &= 5 + 5i + 5i - 5 \\
 &= 10i
 \end{aligned}$$

$$\begin{aligned}
 \text{i } (3-2i)(5+i)(4-2i) &= (3-2i)[5(4-2i) + i(4-2i)] \\
 &= (3-2i)[20 - 10i + 4i - 2i^2] \\
 &= (3-2i)[20 - 10i + 4i + 2] \\
 &= (3-2i)[22 - 6i] \\
 &= 3(22 - 6i) - 2i(22 - 6i) \\
 &= 66 - 18i - 44i + 12i^2 \\
 &= 66 - 18i - 44i - 12 \\
 &= 54 - 62i
 \end{aligned}$$

$$\begin{aligned}
 \text{j } (2+3i)^3 &= (2+3i)[(2+3i)(2+3i)] \\
 &= (2+3i)[2(2+3i) + 3i(2+3i)] \\
 &= (2+3i)[4 + 6i + 6i + 9i^2] \\
 &= (2+3i)[4 + 6i + 6i - 9] \\
 &= (2+3i)(-5 + 12i) \\
 &= 2(-5 + 12i) + 3i(-5 + 12i) \\
 &= -10 + 24i - 15i + 36i^2 \\
 &= -10 + 24i - 15i - 36 \\
 &= -46 + 9i
 \end{aligned}$$

$$\begin{aligned}
 2 \text{ a } (4+5i)(4-5i) &= 16 - 20i + 20i - 25i^2 \\
 &= 16 - 20i + 20i + 25 \\
 &= 41
 \end{aligned}$$

$$\begin{aligned}
 \text{b } (7-2i)(7+2i) &= 49 + 14i - 14i - 4i^2 \\
 &= 49 + 14i - 14i + 4 \\
 &= 53
 \end{aligned}$$

c The answers to **2a** and **2b** are both real.

d Let  $a$  and  $b$  be any real numbers.

$$\begin{aligned}
 (a+bi)(a-bi) &= a^2 - abi + abi - b^2i^2 \\
 &= a^2 - abi + abi + b^2 \\
 &= a^2 + b^2
 \end{aligned}$$

For any  $a$  and  $b$ , the imaginary parts cancel (i.e. they sum to zero), so the answer is always real.

$$3 \quad (a+3i)(1+bi) = 25 - 39i$$

$$a + abi + 3i + 3bi^2 = 25 - 39i$$

$$a + abi + 3i - 3b = 25 - 39i$$

$$(a-3b) + (ab+3)i = 25 - 39i$$

Equating real parts:

$$a - 3b = 25$$

$$a = 3b + 25 \quad (1)$$

Equating imaginary parts:

$$ab + 3 = -39 \quad (2)$$

Substituting (1) into (2):

$$(3b+25)b + 3 = -39$$

$$3b^2 + 25b + 3 = -39$$

$$3b^2 + 25b + 42 = 0$$

$$(3b+7)(b+6) = 0$$

$$\text{So } b = -\frac{7}{3} \text{ or } b = -6$$

Substituting  $b = -6$  into (1):

$$a(-6) + 3 = -39$$

$$a = 7$$

Substituting  $b = -\frac{7}{3}$  into (1):

$$a\left(-\frac{7}{3}\right) + 3 = -39$$

$$a = 18$$

Hence, the two pairs of values are:

$$a = 7, b = -6$$

$$a = 18, b = -\frac{7}{3}$$

$$\begin{aligned}
 4 \text{ a } i \times i \times i \times i \times i \times i &= i^2 \times i^2 \times i^2 \\
 &= -1 \times -1 \times -1 \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 \text{b } 3i \times 3i \times 3i \times 3i &= 81(i \times i \times i \times i) \\
 &= 81(i^2 \times i^2) \\
 &= 81(-1 \times -1) \\
 &= 81
 \end{aligned}$$

$$\begin{aligned}
 \text{c } (i \times i \times i \times i \times i) + i &= (i^2 \times i^2 \times i) + i \\
 &= (-1 \times -1 \times i) + i \\
 &= i + i \\
 &= 2i
 \end{aligned}$$

$$\begin{aligned}
 \text{d } (4i)^3 - 4i^3 &= (4i \times 4i \times 4i) - 4(i \times i \times i) \\
 &= 64(i \times i \times i) - 4(i \times i \times i) \\
 &= 60(i \times i \times i) \\
 &= 60(-1 \times i) \\
 &= -60i
 \end{aligned}$$

5 To expand  $(1+i)^6$ , use the binomial expansion of  $(a+b)^6$

$$\begin{aligned}
 (a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 \\
 &\quad + 15a^2b^4 + 6ab^5 + b^6
 \end{aligned}$$

Substitute  $a = 1$  and  $b = i$ :

$$\begin{aligned}
 (1+i)^6 &= (1)^6 + 6(1)^5(i) + 15(1)^4(i)^2 + 20(1)^3(i)^3 \\
 &\quad + 15(1)^2(i)^4 + 6(1)(i)^5 + (i)^6
 \end{aligned}$$

$$(1+i)^6 = 1 + 6i + 15i^2 + 20i^3 + 15i^4 + 6i^5 + i^6$$

[Use  $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$  and  $i^6 = -1$ ]

$$\begin{aligned}
 (1+i)^6 &= 1 + 6i - 15 - 20i + 15 + 6i - 1 \\
 &= -8i
 \end{aligned}$$

So  $a = 0$  and  $b = -8$

- 6 To expand  $(3-2i)^4$ , use the binomial expansion of  $(a+b)^4$ :

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Substitute  $a = 3$  and  $b = 2i$ :

$$(3-2i)^4 = (3)^4 + 4(3)^3(-2i) + 6(3)^2(-2i)^2 + 4(3)(-2i)^3 + (-2i)^4$$

$$(3-2i)^4 = 81 - 216i + 216i^2 - 96i^3 + 16i^4$$

[Use  $i^2 = -1$ ,  $i^3 = -i$  and  $i^4 = 1$ ]

$$(3-2i)^4 = 81 - 216i - 216 + 96i + 16 = -119 - 120i$$

So the real part of  $(3-2i)^4$  is  $-119$

- 7 a  $f(2i) = 2(2i)^2 - (2i) + 8$   
 $= 2(4i^2) - 2i + 8$   
 $= 8(-1) - 2i + 8$   
 $= -2i$

b  $f(3-6i)$   
 $= 2(3-6i)^2 - (3-6i) + 8$   
 $= 2(9-36i+36i^2) - 3 + 6i + 8$   
 $= 18 - 72i + 72i^2 - 3 + 6i + 8$   
 $= 18 - 72i - 72 - 3 + 6i + 8$   
 $= -49 - 66i$

- 8  $f(1-4i) = (1-4i)^2 - 2(1-4i) + 17$   
 $= 1 - 8i + 16i^2 - 2 + 8i + 17$   
 $= 1 - 8i - 16 - 2 + 8i + 17$   
 $= 0$

$f(1-4i) = 0 \Rightarrow z = 1-4i$  is a solution of

$$f(z) = 0$$

- 9 a  $i^3 = (i^2)(i) = (-1)(i) = -i$   
 $i^4 = (i^2)(i^2) = (-1)(-1) = 1$

- 9 b  $i^5 = (i^4)(i) = (1)(i) = i$   
 $i^6 = (i^4)(i^2) = (1)(-1) = -1$   
 $i^7 = (i^4)(i^3) = (1)(-i) = -i$   
 $i^8 = (i^4)(i^4) = (1)(1) = 1$

- c For  $n \in \mathbb{N}$ ,

$$i^{4n} = 1, i^{4n+1} = i, i^{4n+2} = -1, i^{4n+3} = -i$$

Hence:

i  $100 = 4 \times 25 \Rightarrow i^{100} = 1$

ii  $253 = (4 \times 63) + 1 \Rightarrow i^{253} = i$

iii  $301 = (4 \times 75) + 1 \Rightarrow i^{301} = i$

### Challenge

a  $(a+bi)^2 = (a+bi)(a+bi)$   
 $= a^2 + abi + abi + b^2i^2$   
 $= a^2 + 2abi + b^2(-1)$   
 $= (a^2 - b^2) + 2abi$

b Let  $a^2 - b^2 = 40$  (1)  
and  $2ab = -42$  (2)

Then by part a,  $\sqrt{40-42i} = a+bi$

Rearranging (2) gives  $a = -\frac{21}{b}$  (3)

Substituting (3) into (1) gives:

$$\left(-\frac{21}{b}\right)^2 - b^2 = 40$$

$$\frac{441}{b^2} - b^2 = 40$$

$$b^4 + 40b^2 - 441 = 0$$

$$(b^2 - 9)(b^2 + 49) = 0$$

$b$  is real, so  $b^2 \neq -49$

Hence  $b^2 - 9 = 0 \Rightarrow b = -3$

Substituting  $b = -3$  into (3):

$$a = -\frac{21}{b} = \frac{-21}{-3} = 7$$

So  $\sqrt{40-42i} = 7-3i$