FP1 2017 (MA)

Q1a) $f(6) = -0.8888\ldots$
$f(7) = 1.414966\ldots$

Change in sign between $x=6$ and $x=7$ $\therefore$ a root lies between these values.

b) $f(x) = \frac{1}{3}x^2 + 4x^{-2} - 2x - 1$

$f'(x) = \frac{2}{3}x - 8x^{-3} - 2$

$f(6) = -0.8888\ldots$
$f'(6) = \frac{2}{3}(6) - 8(6)^{-3} - 2 = 1.96296\ldots$

$\therefore x_1 = 6 - \frac{-0.8888\ldots}{1.96296\ldots} = 6.45283\ldots$
$= \boxed{6.45}$

Q2a) $\det A = 6+4 = 10$

$\therefore A^{-1} = \boxed{\frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}}$

b) $P = AB$

$A^{-1}P = A^{-1}AB$

$\therefore A^{-1}P = B = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 11 & -8 \end{pmatrix}$

$$= \frac{1}{10}\begin{pmatrix} 9+11 & 18-8 \\ 22-12 & -16-24 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 20 & 10 \\ 10 & -40 \end{pmatrix}$$

$$B = \boxed{\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}}$$

Q3a) $P(1, 16)$ $\quad Q(8, 2)$

$$m_{PQ} = \frac{16-2}{1-8} = \frac{14}{-7} = -2$$

$\therefore \boxed{y = (1/2)x}$ (passes through o)

b) $xy = 4t \times \frac{4}{t}$

$\boxed{xy = 16}$

c) $y = \left(\frac{1}{2}\right)x$

$\hookrightarrow x\left(\frac{1}{2}x\right) = 16$

$\Rightarrow x^2 = 32 \quad \therefore x = \pm 4\sqrt{2} \; (= \sqrt{32})$

$\therefore y = \pm 2\sqrt{2}$ so points are $\boxed{(4\sqrt{2}, 2\sqrt{2})}$

and $\boxed{(-4\sqrt{2}, -2\sqrt{2})}$

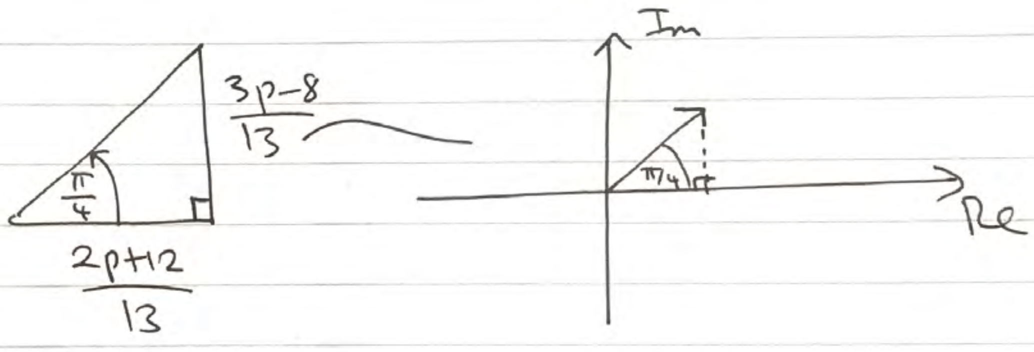
Q4i a) $w = \frac{(p-4i)}{(2-3i)} \frac{\times(2+3i)}{\times(2+3i)}$

$$= \frac{(p-4i)(2+3i)}{4 - \cancel{6}i + \cancel{6}i + 9} = \frac{2p + 3pi - 8i - 12i^2}{13}$$

$$= \frac{(2p+12) + (3p-8)i}{13}$$

$$= \left(\frac{2p+12}{13}\right) + \left(\frac{3p-8}{13}\right)i$$

b)



$$\Rightarrow \tan\frac{\pi}{4} = \frac{\frac{3p-8}{\cancel{13}}}{\frac{2p+12}{\cancel{13}}} = \frac{3p-8}{2p+12} = 1$$

$$\Rightarrow 2p + 12 = 3p - 8$$

$$\Rightarrow \boxed{p = 20}$$

ii) $z = 4 + 3i - 4\lambda i + 3\lambda$

$z = (4+3\lambda) + (3-4\lambda)i$

$|z| = \sqrt{(4+3\lambda)^2 + (3-4\lambda)^2} = 45$

$\therefore \quad (4+3\lambda)^2 + (3-4\lambda)^2 = 45^2$

$\Rightarrow \quad 16 + \cancel{24\lambda} + 9\lambda^2 + 9 - \cancel{24\lambda} + 16\lambda^2 = 2025$

$\Rightarrow \quad 25\lambda^2 + 25 = 2025$

$\Rightarrow \quad 25\lambda^2 = 2000$

$\Rightarrow \quad \lambda^2 = 80$

$\Rightarrow \quad \lambda = \boxed{4\sqrt{5}} \ , \quad \lambda = \boxed{-4\sqrt{5}}$

Q5 i a) $AB = \begin{pmatrix} p & 2 \\ 3 & p \end{pmatrix}\begin{pmatrix} -5 & 4 \\ 6 & -5 \end{pmatrix} = \boxed{\begin{pmatrix} -5p+12 & 4p-10 \\ -15+6p & -5p+12 \end{pmatrix}}$

b) $AB + 2A = kI$

$$\begin{pmatrix} 12-5p & 4p-10 \\ 6p-15 & 12-5p \end{pmatrix} + \begin{pmatrix} 2p & 4 \\ 6 & 2p \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$\Rightarrow \quad 6p - 15 + 6 = 0$

$\therefore \quad 6p = 9 \quad \therefore p = \boxed{\frac{3}{2}}$ //

$\Rightarrow \quad 12 - 5p + 2p = k$

$12 - 3\left(\frac{3}{2}\right) = k = \boxed{\frac{15}{2}}$ //

ii) $\text{Area}_{\text{image}} = \text{Area}_{\text{object}} \times \det|M|.$

$270 = 15 \det M$

$\therefore \quad \frac{270}{15} = |\det M| = 18$ //

$|\det M| = |2a+9| = 18$

$2a + 9 = 18 \quad$ or $\quad -2a - 9 = 18$

$\boxed{a = 9/2} \qquad \boxed{a = -27/2}$

Q6a) $-3-2i$ (complex conjugate pair)

b) $(x-(-3-2i))(x-(-3+2i))(x-4)=0$

$\Rightarrow (x^2-x(-6)+(9+4))(x-4)=0$

$\Rightarrow (x^2+6x+13)(x-4)=0$

$\Rightarrow x^3+6x^2+13x-4x^2-24x-52=0$

$\Rightarrow x^3+2x^2-11x-52=0$

$\therefore \boxed{a=2,\ b=-11}$

Q7a) $y^2=4ax$ [Implicit differentiation]

$2y\frac{dy}{dx}=4a \quad \therefore \frac{dy}{dx}=\frac{2a}{y}=\frac{2a}{2aq}=\frac{1}{q}$ //

$\Rightarrow y-2aq=\frac{1}{q}(x-aq^2)$

$\Rightarrow y=\frac{1}{q}x-aq+2aq$

$\xRightarrow{\times q} qy=x+aq^2$

$\square$

b) <u>substitute X into tangent:</u>

$$q(0) = -\frac{1}{4}a + aq^2$$

$$\frac{a}{4} = aq^2$$

$$1 = 4q^2$$

$$\therefore q^2 = \frac{1}{4} \text{ so } q = \frac{1}{2}$$

<u>directrix : $x = -a$</u> : $qy = -a + aq^2$

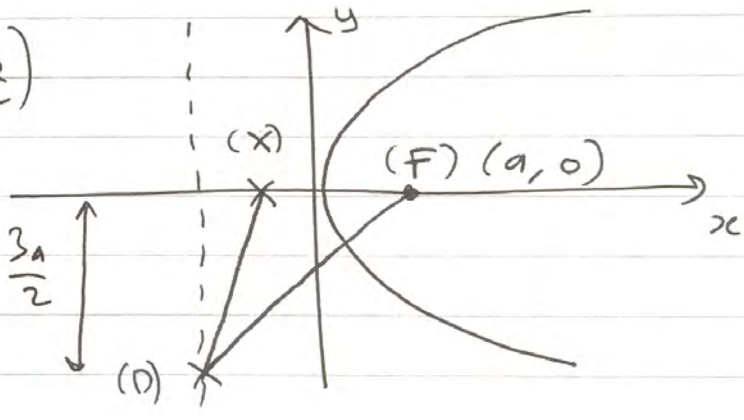
$$\frac{1}{2}y = -a + \frac{1}{4}a$$

$$y = -\frac{3a}{2}$$

$$\therefore \boxed{D\left(-a, -\frac{3a}{2}\right)}$$

c) $\text{Area} = \frac{1}{2}\underbrace{\left(a + \frac{a}{4}\right)}_{XF}\underbrace{\left(\frac{3a}{2}\right)}_{y_D}$

$$= \boxed{\frac{15a^2}{16}}$$



Q8a) $3\sum_{1}^{n}(r^2) + 8\sum_{1}^{n}(r) + 3\sum_{1}^{n}(1)$

$= \frac{3n}{6}(n+1)(2n+1) + \frac{8n}{2}(n+1) + 3n$

$= \frac{n}{2}(2n^2+3n+1) + 4n^2 + 4n + 3n$

$= n^3 + \frac{3n^2}{2} + \frac{n}{2} + 4n^2 + 7n$

$= n^3 + \frac{11}{2}n^2 + \frac{15n}{2} = \frac{1}{2}(2n^3 + 11n^2 + 15n)$

$= \frac{n}{2}(2n^2 + 11n + 15) = \frac{n}{2}(2n+5)(n+3)$

$\square$

b) $\sum_{1}^{12}(3r^2+8r+3) + k\sum_{1}^{12}2^{r-1} = 3520$

$\frac{12}{2}(24+5)(15) + k\sum_{1}^{12}\frac{2^r}{2^1} = 3520$

$\frac{k}{2}\sum_{1}^{12}2^r = 910$

$k\sum_{1}^{12}2^r = 1820$

$\sum_{1}^{12}2^r$ ~ Geometric series from (2, ...)

$2 + 2(2) + 2(2)^2 + \ldots$

$a = 2,\ r = 2$

$$\therefore \sum_{1}^{12} 2^r = \frac{a(1-r^n)}{1-r} = \frac{2(1-(2)^{12})}{1-2} = 8190$$

$$\Rightarrow \quad 8190(k) = 1820$$

$$\Rightarrow \quad k = \frac{1820}{8190} = \boxed{\frac{2}{9}}$$

Q9 i) $u_1 = 6$ $\quad u_{n+2} = 6u_{n+1} - 9u_n$.

$u_2 = 27$ $\quad$ prove: $u_n = 3^n(n+1)$

$n=1$: $\quad u_1 = 3^1(2) = 6$ as given.

$n=2$: $\quad u_2 = 3^2(3) = 27$ as given.

$\therefore$ true for $n=1$, $n=2$.

now assume $[u_k = 3^k(k+1)]$ is true,

and $[u_{k+1} = 3^{k+1}(k+2)]$ is true

consider $n = k+2$,

$$\Rightarrow \quad u_{k+2} = 6u_{k+1} - 9u_k$$

$$= 6\cdot 3^{k+1}(k+2) - 3^2\cdot 3^k(k+1)$$

$$= 2\cdot 3^{k+2}(k+2) - 3^{k+2}(k+1)$$

$$= 2\cdot 3^{k+2}(k+1) + 2\cdot 3^{k+2} - 3^{k+2}(k+1)$$

$$= (2-1)3^{k+2}(k+1) + 2(3^{k+2})$$

$= 3^{k+2}(k+1) + 2 \cdot 3^{k+2}$

$= 3^{k+2}(k+3)$

$\therefore$ true for $n=k+2$

so, true for $n=1,2$.
true for $n=k+2$ when true for $n=k, n=k+1$.
$\therefore$ By Mathematical Induction the relationship is true for all $n \in \mathbb{Z}^+$

ii) $f(n) = 3^{3n-2} + 2^{3n+1}$

$\underline{n=1}$: $f(1) = 3 + 2^4 = 19 = (19) \times 1$
$\therefore$ true for $n=1$.

assume $f(k)$ is divisible by 19,

consider $f(k+1)$,

$f(k+1) = 3^{3(k+1)-2} + 2^{3(k+1)+1}$

$= 3^{3k+1} + 2^{3k+4}$

$= 3^{3k-2+3} + 2^{3k+1+3}$

$= 3^3(3^{3k-2}) + 8(2^{3k+1})$

$= 27(3^{3k-2}) + 8(2^{3k+1})$

[factoring out f(k)]

$$= 27\left[3^{3k-2} + 2^{3k+1}\right] - 19\left(2^{3k+1}\right)$$

$$= 27f(k) - 19\left(2^{3k+1}\right)$$

$\therefore$ true for $n = k+1$.

so, true for $n = 1$.
true for $n = k+1$ when assumed true for $n = k$.

$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$