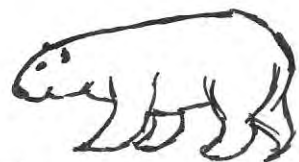


1. Given that k is a real number and that

$$A = \begin{pmatrix} 1+k & k \\ k & 1-k \end{pmatrix}$$



find the exact values of k for which A is a singular matrix. Give your answers in their simplest form.

(3)

if $\text{Det}(A) = 0 \Rightarrow \text{Singular}$

$$\text{Det}(A) = (1+k)(1-k) - k^2 = 1 - 2k^2$$

$$\therefore \text{If singular } 2k^2 = 1 \quad \therefore k^2 = \frac{1}{2} \quad \therefore k = \pm \sqrt{\frac{1}{2}}$$

$$k = \pm \frac{\sqrt{2}}{2}$$

2. $f(x) = 3x^{\frac{3}{2}} - 25x^{-\frac{1}{2}} - 125, \quad x > 0$

(a) Find $f'(x)$.

(2)

The equation $f(x) = 0$ has a root α in the interval $[12, 13]$.

(b) Using $x_0 = 12.5$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 decimal places.

(4)

$$a) f'(x) = \frac{9}{2}x^{\frac{1}{2}} + \frac{25}{2}x^{-\frac{3}{2}}$$

$$b) x_0 = 12.5 \Rightarrow x_1 = 12.5 - \frac{f(12.5)}{f'(12.5)}$$

$$x_1 = 12.5 - \frac{3(12.5)^{\frac{3}{2}} - 25(12.5)^{-\frac{1}{2}} - 125}{\frac{9}{2}(12.5)^{\frac{1}{2}} + \frac{25}{2}(12.5)^{-\frac{3}{2}}}$$

$$\therefore x_1 = 12.468$$

3. (a) Using the formula for $\sum_{r=1}^n r^2$ write down, in terms of n only, an expression for

$$\sum_{r=1}^{3n} r^2$$

(1)

- (b) Show that, for all integers n , where $n > 0$

$$\sum_{r=2n+1}^{3n} r^2 = \frac{n}{6}(an^2 + bn + c)$$

where the values of the constants a , b and c are to be found.

(4)

$$a) \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\Rightarrow \sum_{r=1}^{3n} r^2 = \frac{1}{6}(3n)(3n+1)(6n+1)$$

$$b) \sum_{r=2n+1}^{3n} r^2 = \sum_{r=1}^{3n} r^2 - \sum_{r=1}^{2n} r^2$$

$$= \frac{1}{6}(3n)(3n+1)(6n+1) - \frac{1}{6}(2n)(2n+1)(4n+1)$$

$$= \frac{1}{6}n [(9n+3)(6n+1) - (4n+2)(4n+1)]$$

$$= \frac{1}{6}n [54n^2 + 27n + 3 - 16n^2 - 12n - 2]$$

$$= \frac{1}{6}n [38n^2 + 15n + 1] \quad a=38 \quad b=15 \quad c=1$$

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4.

$$z = \frac{4}{1+i}$$

Find, in the form $a + ib$ where $a, b \in \mathbb{R}$

(a) z

(2)

(b) z^2

(2)

Given that z is a complex root of the quadratic equation $x^2 + px + q = 0$, where p and q are real integers,

(c) find the value of p and the value of q .

(3)

$$a) \quad z = \frac{4(1-i)}{(1+i)(1-i)} = \frac{4-4i}{1-i^2} = \frac{4-4i}{2} = \underline{2-2i}$$

$$b) \quad z^2 = (2-2i)^2 = 4 - 8i + 4i^2 = \underline{-8i}$$

$$c) \quad \text{roots } \alpha, \beta \quad \alpha = 2-2i \Rightarrow \beta = 2+2i$$

$$x^2 - (\alpha + \beta)x + (\alpha\beta)$$

$$x^2 - 4x + (4 - 4i^2) \Rightarrow x^2 - 4x + 8 \quad \begin{array}{l} p = -4 \\ q = 8 \end{array}$$

5. Points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p^2 \neq q^2$, lie on the parabola $y^2 = 4ax$.

(a) Show that the chord PQ has equation

$$y(p + q) = 2x + 2apq \tag{5}$$

Given that this chord passes through the focus of the parabola,

(b) show that $pq = -1$ (1)

(c) Using calculus find the gradient of the tangent to the parabola at P . (2)

(d) Show that the tangent to the parabola at P and the tangent to the parabola at Q are perpendicular. (2)

$$a) \quad M_{PQ} = \frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2a(q-p)}{a(q-p)(q+p)} = \frac{2}{q+p}$$

$$y - y_1 = m(x - x_1) \Rightarrow y - 2ap = \frac{2}{q+p}(x - ap^2)$$

$$\Rightarrow y(q+p) - 2ap(q+p) = 2x - 2ap^2$$

$$\Rightarrow y(q+p) = 2x - 2ap^2 + 2apq + 2ap^2$$

$$\therefore y(p+q) = 2x + 2apq \quad \#$$

b) chord passes through focus $(a, 0)$

$$\Rightarrow 0(p+q) = 2a + 2apq$$

$$\Rightarrow -2a = 2apq \quad \therefore pq = \frac{-2a}{2a} \quad \therefore pq = -1 \quad \#$$

$$c) y^2 = 4ax \Rightarrow \frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

$$\Rightarrow 2y \frac{dy}{dx} = 4a \quad \therefore \frac{dy}{dx} = \frac{4a}{2y}$$

$$\text{at } P \ y = 2ap \quad \therefore M_t = \frac{4a}{2(2ap)} = \frac{4a}{4ap} = \frac{1}{p}.$$

$$d) \text{ if perp } M_{t_p} \times M_{t_q} = -1$$

$$M_{t_p} = \frac{1}{p} \quad M_{t_q} = \frac{1}{q}$$

$$M_{t_p} \times M_{t_q} = \frac{1}{p} + \frac{1}{q} = \frac{1}{pq} = \frac{1}{-1} = -1 \quad \#$$

6.

$$\mathbf{P} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single geometrical transformation U represented by the matrix \mathbf{P} . (2)

The transformation U maps the point A , with coordinates (p, q) , onto the point B , with coordinates $(6\sqrt{2}, 3\sqrt{2})$.

(b) Find the value of p and the value of q . (3)

The transformation V , represented by the 2×2 matrix \mathbf{Q} , is a reflection in the line with equation $y = x$.


(c) Write down the matrix \mathbf{Q} . (1)

The transformation U followed by the transformation V is the transformation T . The transformation T is represented by the matrix \mathbf{R} .

(d) Find the matrix \mathbf{R} . (3)

(e) Deduce that the transformation T is self-inverse. (1)

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



rotation about $(0,0)$ through 135° .

b) $PA = B \Rightarrow A = P^{-1}(B)$

$$P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \det(P) = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

$$\therefore P^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6\sqrt{2} \\ 3\sqrt{2} \end{pmatrix} = \begin{pmatrix} -6+3 \\ -6-3 \end{pmatrix} = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ reflection through $y=x$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
 y becomes x , x becomes y

d) $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = R$

e) $\det(R) = \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} - \frac{1}{2} = -1$

$$R^{-1} = -1 \begin{pmatrix} -\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{2}} & +\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\therefore R = R^{-1} \therefore$ Self Inverse
 ?

7. A complex number z is given by

$$z = a + 2i$$

where a is a non-zero real number.

(a) Find $z^2 + 2z$ in the form $x + iy$ where x and y are real expressions in terms of a . (4)

Given that $z^2 + 2z$ is real,

(b) find the value of a . (1)

Using this value for a ,

(c) find the values of the modulus and argument of z , giving the argument in radians, and giving your answers to 3 significant figures. (3)

(d) Show the points P , Q and R , representing the complex numbers z , z^2 and $z^2 + 2z$ respectively, on a single Argand diagram with origin O . (3)

(e) Describe fully the geometrical relationship between the line segments OP and OR . (2)

$$a) z^2 = (a+2i)^2 = a^2 + 4ai + 4i^2 = (a^2-4) + 4ai$$

$$z^2 + 2z = (a^2-4) + 4ai + 2a + 4i$$

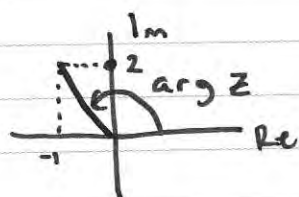
$$= (a^2+2a-4) + (4a+4)i$$

$$\begin{aligned} x &= a^2+2a-4 \\ y &= 4a+4 \end{aligned}$$

$$b) z^2 + 2z \text{ is Real} \Rightarrow \text{Im} = 0 \quad \therefore 4a+4 = 0$$

$$a = -1$$

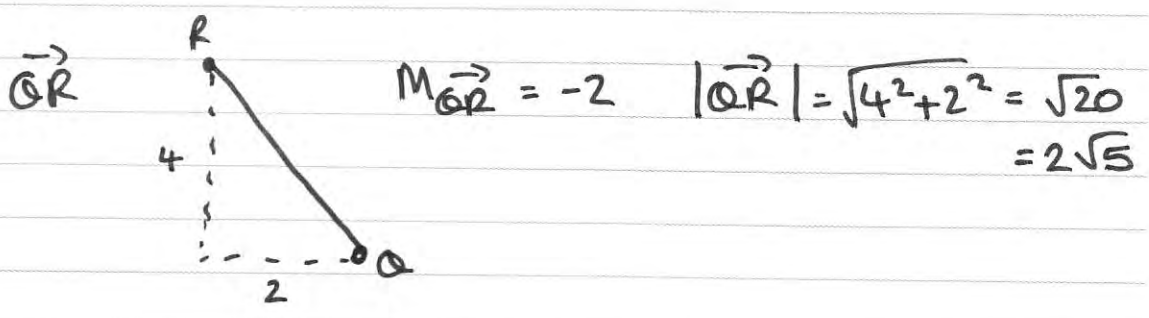
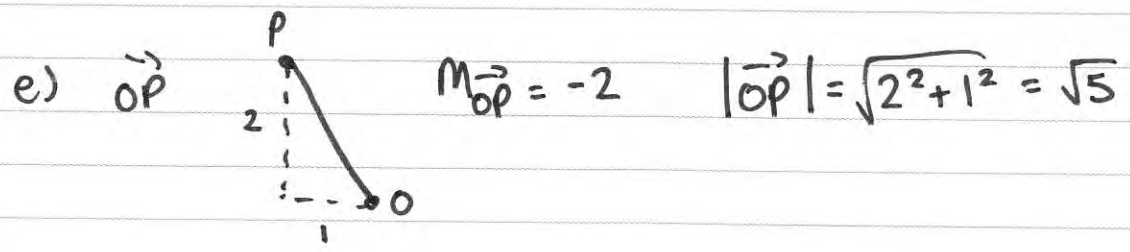
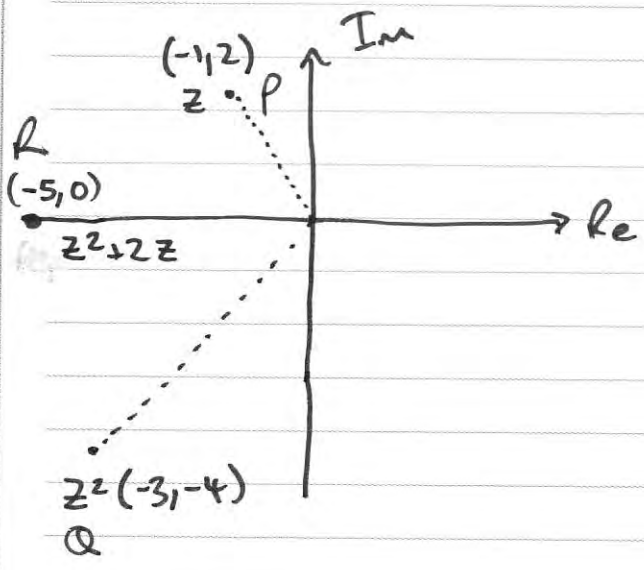
$$c) |z| = |-1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$$



$$\arg z = \pi - \tan^{-1}\left(\frac{2}{1}\right)$$

$$= 2.03^\circ$$

d) $z = -1 + 2i$ $z^2 = -3 - 4i$ $z^2 + 2z = -5$



\vec{OP} and \vec{QR} are parallel
 \vec{QR} is twice the length of \vec{OP}
 $\vec{QR} = 2 \times \vec{OP}$

8. (i) Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^n \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2} \quad (5)$$

(ii) A sequence of positive rational numbers is defined by

$$u_1 = 3$$

$$u_{n+1} = \frac{1}{3}u_n + \frac{8}{9}, \quad n \in \mathbb{Z}^+$$

Prove by induction that, for $n \in \mathbb{Z}^+$

$$u_n = 5 \times \left(\frac{1}{3}\right)^n + \frac{4}{3} \quad (5)$$

$$a) \quad n=1 \quad \sum_1^1 \frac{2r+1}{r^2(r+1)^2} = \frac{3}{1(2)^2} = \frac{3}{4}$$

$$1 - \frac{1}{(1+1)^2} = 1 - \frac{1}{4} = \frac{3}{4} \quad \therefore \text{LHS} = \text{RHS} \text{ true for } n=1$$

assume true for $n=k$.

$$\sum_1^k \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(k+1)^2}$$

$$n=k+1 \quad \sum_1^{k+1} \frac{2r+1}{r^2(r+1)^2} = \sum_1^k \frac{2r+1}{r^2(r+1)^2} + \frac{2(k+1)+1}{(k+1)^2(k+2)^2}$$

$$= \left(1 - \frac{1}{(k+1)^2}\right) + \frac{2k+3}{(k+1)^2(k+2)^2}$$

$$= 1 - \left(\frac{1}{(k+1)^2(k+2)^2} - \frac{2k+3}{(k+1)^2(k+2)^2} \right)$$

$$= 1 - \frac{u^2 + 4u + 4 - 2u - 3}{(u+1)^2(u+2)^2} = 1 - \frac{u^2 + 2u + 1}{(u+1)^2(u+2)^2}$$

$$= 1 - \frac{(u+1)^2}{(u+1)^2(u+2)^2} = 1 - \frac{1}{(u+2)^2}$$

$$n = u+1 \quad \text{RHS} = 1 - \frac{1}{(u+1+1)^2} = 1 - \frac{1}{(u+2)^2}$$

$\therefore \text{LHS} = \text{RHS} \quad \therefore \text{true for } n = u+1$

true for $n=1$, if true for $n=k$ then true for $n=k+1$
 \therefore by Mathematical Induction true for all $n \in \mathbb{Z}^+$

$$(ii) \quad u_1 = 3 \quad u_1 = 5\left(\frac{1}{3}\right)^1 + \frac{4}{3} = \frac{5}{3} + \frac{4}{3} = 3$$

$\text{LHS} = \text{RHS} \quad \therefore \text{true for } n=1$

$$u_2 = \frac{1}{3}(3) + \frac{8}{9} = \frac{17}{9}$$

$$u_2 = 5\left(\frac{1}{3}\right)^2 + \frac{4}{3} = \frac{5}{9} + \frac{12}{9} = \frac{17}{9}$$

$\text{LHS} = \text{RHS} \quad \therefore \text{true for } n=2$

assume true for $n=k \Rightarrow u_k = 5\left(\frac{1}{3}\right)^k + \frac{4}{3}$

$$\begin{aligned} n = k+1 \quad u_{k+1} &= \frac{1}{3}u_k + \frac{8}{9} = \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{4}{9} + \frac{8}{9} \\ &= \frac{5}{3}\left(\frac{1}{3}\right)^k + \frac{12}{9} = 5\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^k + \frac{4}{3} \\ &= 5\left(\frac{1}{3}\right)^{k+1} + \frac{4}{3} \end{aligned}$$

$$\text{RHS} \quad u_{u+1} = 5 \left(\frac{1}{3} \right)^{u+1} + \frac{4}{3}$$

$\therefore \text{RHS} = \text{LHS} \quad \therefore \text{true for } n = u+1$

true for $n=1, n=2$ true for $n=u+1$ if true for $n=u$ \therefore by Mathematical Induction true for all $n \in \mathbb{Z}^+$.

9. The rectangular hyperbola, H , has cartesian equation $xy = 25$

(a) Show that an equation of the normal to H at the point $P\left(5p, \frac{5}{p}\right)$, $p \neq 0$, is

$$y - p^2x = \frac{5}{p} - 5p^3 \quad (5)$$

This normal meets the line with equation $y = -x$ at the point A .

(b) Show that the coordinates of A are

$$\left(-\frac{5}{p} + 5p, \frac{5}{p} - 5p\right) \quad (3)$$

The point M is the midpoint of the line segment AP .

Given that M lies on the positive x -axis,

(c) find the exact value of the x coordinate of point M .

(3)

$$a) \quad y = 25x^{-1} \Rightarrow \frac{dy}{dx} = -25x^{-2} = -\frac{25}{x^2}$$

$$\Rightarrow \text{at } P \quad M_t = \frac{-25}{25p^2} = -\frac{1}{p^2}$$

$$M_n = p^2$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{5}{p} = p^2(x - 5p) \Rightarrow y - \frac{5}{p} = p^2x - 5p^3$$

$$\therefore y - p^2x = \frac{5}{p} - 5p^3$$

$$b) \quad y = -x \Rightarrow -x - p^2x = \frac{5}{p} - 5p^3$$

$$\Rightarrow x + p^2x = 5p^3 - \frac{5}{p} = \frac{5p^4 - 5}{p}$$

$$\Rightarrow x(1 + p^2) = \frac{5(p^4 - 1)}{p}$$

$$\Rightarrow x = \frac{5(p^2+1)(p^2-1)}{p(p^2+1)} = \frac{5p^2 - 5}{p}$$

$$\therefore x = 5p - \frac{5}{p} \quad y = \frac{5}{p} - 5p \quad \left(5p - \frac{5}{p}, \frac{5}{p} - 5p\right) \quad \text{PMT}$$

c) Midpoint AP

$$= \left(\frac{5p - \frac{5}{p} + 5p}{2}, \frac{\frac{5}{p} + \frac{5}{p} - 5p}{2} \right)$$
$$= \left(5p - \frac{5}{2p}, \frac{5}{p} - \frac{5p}{2} \right)$$

M lies on x-axis $\therefore y = 0$

$$\frac{5}{p} = \frac{5p}{2} \Rightarrow 10 = 5p^2 \therefore p^2 = 2$$
$$\therefore p = \sqrt{2}$$

(not $-\sqrt{2}$ as it is on the x-axis)

$$x = 5\sqrt{2} - \frac{5}{2\sqrt{2}} = 5\sqrt{2} - \frac{5\sqrt{2}}{4} = \frac{15\sqrt{2}}{4}$$

2