

FPI SIS

1.

$$f(x) = 9x^3 - 33x^2 - 55x - 25$$



Given that $x=5$ is a solution of the equation $f(x)=0$, use an algebraic method to solve $f(x)=0$ completely.

(5)

$(x-5)$ is a factor

$$f(x) = (x-5)(9x^2 + Ax + 5)$$

$$\Rightarrow -45x^2 + Ax^2 \equiv -33x^2 \quad \therefore A = 12$$

$$f(x) = (x-5)(9x^2 + 12x + 5)$$

$$\Rightarrow f(x) = 0 \Rightarrow x = 5 \quad 9x^2 + 12x + 5 = 0$$

$$x^2 + \frac{4}{3}x + \frac{5}{9} = 0$$

$$\Rightarrow \left(x + \frac{2}{3}\right)^2 = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}$$

$$\therefore x + \frac{2}{3} = \pm \sqrt{-\frac{1}{9}} = \pm \frac{1}{3}i \quad \therefore x = -\frac{2}{3} \pm \frac{1}{3}i$$

$$\text{If } f(x) = 0 \quad x = 5, -\frac{2}{3} + \frac{1}{3}i, -\frac{2}{3} - \frac{1}{3}i$$



2. In the interval $13 < x < 14$, the equation

$$3 + x \sin\left(\frac{x}{4}\right) = 0, \text{ where } x \text{ is measured in radians,}$$

has exactly one root, α .

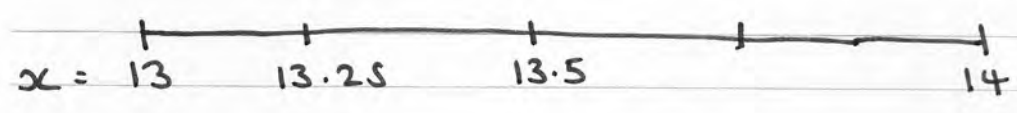
(a) Starting with the interval $[13, 14]$, use interval bisection twice to find an interval of width 0.25 which contains α .

(3)

(b) Use linear interpolation once on the interval $[13, 14]$ to find an approximate value for α . Give your answer to 3 decimal places.

(4)

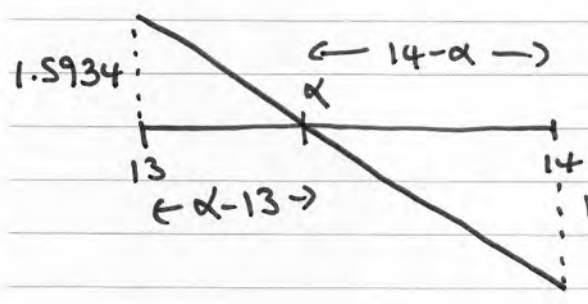
$$f(x) = 3 + x \sin\left(\frac{x}{4}\right)$$



$f(x) =$	1.59	0.75	-0.12	-1.91
	> 0	> 0	< 0	< 0

$\underbrace{\hspace{10em}}_{\text{root } \alpha} \quad \therefore \alpha \in (13.25, 13.50)$

b)



$$\frac{14 - \alpha}{1.9109} = \frac{\alpha - 13}{1.5934}$$

$$22.3076 - 1.5934\alpha$$

$$1.9109\alpha - 24.8417$$

$$\therefore 3.5043\alpha = 47.1493 \quad \therefore \alpha = 13.455$$



3. (a) Using the formulae for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$, show that

$$\sum_{r=1}^n (r+1)(r+4) = \frac{n}{3}(n+4)(n+5)$$

for all positive integers n .

(5)

- (b) Hence show that

$$\sum_{r=n+1}^{2n} (r+1)(r+4) = \frac{n}{3}(n+1)(an+b)$$

where a and b are integers to be found.

(3)

$$a) \sum (r+1)(r+4) = \sum r^2 + 5\sum r + 4\sum 1$$

$$= \frac{1}{6}n(2n^2+3n+1) + 5 \times \frac{1}{2}n(n+1) + 4n$$

$$= \frac{1}{6}n(2n^2+3n+1) + \frac{15}{6}n(n+1) + \frac{24}{6}n$$

$$= \frac{1}{6}n[2n^2+3n+1+15n+15+24]$$

$$= \frac{1}{6}n[2n^2+18n+40]$$

$$= \frac{1}{3}n(n^2+9n+20) = \frac{1}{3}n(n+4)(n+5)$$

$$b) \sum_{r=n+1}^{2n} (r+1)(r+4) = \frac{1}{3}(2n)(2n+4)(2n+5) - \frac{1}{3}n(n+4)(n+5)$$

$$= \frac{1}{3}n[(4n+8)(2n+5) - (n+4)(n+5)]$$

$$= \frac{1}{3}n[8n^2+36n+40 - n^2-9n-20]$$

$$= \frac{1}{3}n[7n^2+27n+20]$$

$$= \frac{1}{3}n(7n+20)(n+1)$$



4.

$$z_1 = 3i \text{ and } z_2 = \frac{6}{1+i\sqrt{3}}$$

- (a) Express z_2 in the form $a + ib$, where a and b are real numbers. (2)
- (b) Find the modulus and the argument of z_2 , giving the argument in radians in terms of π . (4)
- (c) Show the three points representing z_1 , z_2 and $(z_1 + z_2)$ respectively, on a single Argand diagram. (2)

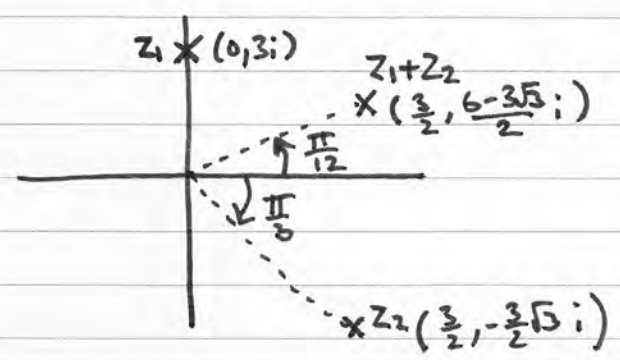
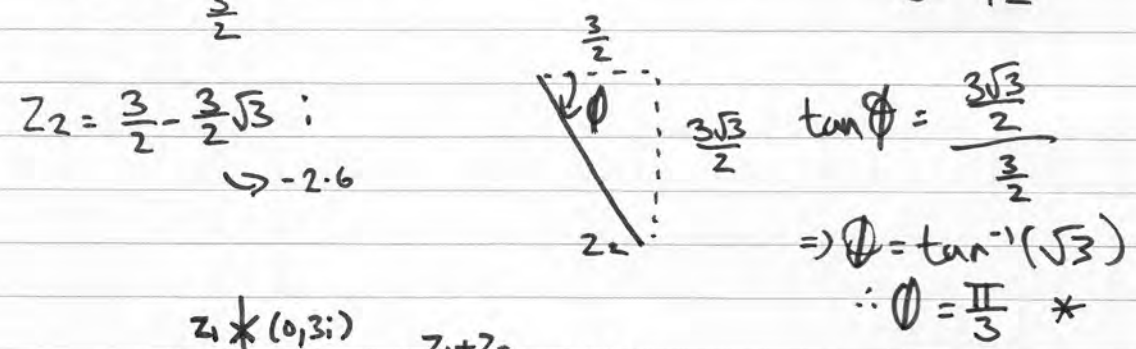
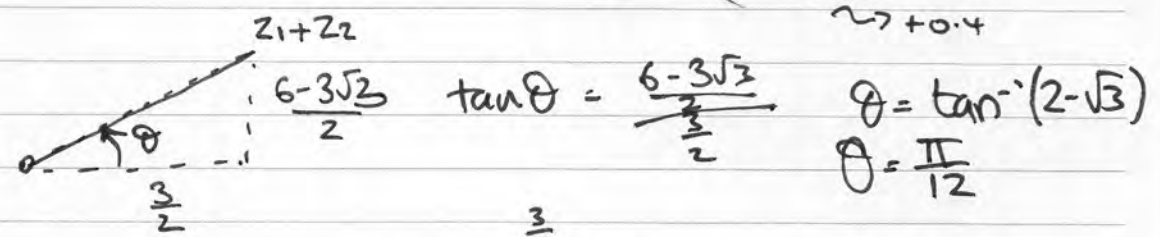
$$a) z_2 = \frac{6(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{6-6i\sqrt{3}}{1+3} = \frac{3}{2} - \frac{3i\sqrt{3}}{2}$$

$$b) |z_2| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$$

$* \arg(z_2) = -\frac{\pi}{3}$

$$c) z_1 + z_2 = 3i + \frac{6}{1+i\sqrt{3}} = 3i + \frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$= \frac{3}{2} + \left(\frac{6-3\sqrt{3}}{2}\right)i$$



Note
angles not required



5. The rectangular hyperbola H has equation $xy = 9$

The point A on H has coordinates $\left(6, \frac{3}{2}\right)$.

(a) Show that the normal to H at the point A has equation

$$2y - 8x + 45 = 0 \quad (5)$$

The normal at A meets H again at the point B .

(b) Find the coordinates of B .

(4)

$$xy = 9 \quad y = \frac{9}{x} = 9x^{-1}$$

$$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2} \quad \Big|_{x=6} \quad \frac{dy}{dx} = -\frac{9}{36} = -\frac{1}{4}$$

$$\therefore m_t = -\frac{1}{4} \Rightarrow m_n = 4 \quad \left(6, \frac{3}{2}\right)$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{3}{2} = 4(x - 6)$$

$$\Rightarrow y - \frac{3}{2} = 4x - 24 \Rightarrow 2y - 3 = 8x - 48$$

$$\therefore 2y - 8x + 45 = 0 \quad \#$$

$$y = \frac{9}{x} \Rightarrow 2y = \frac{18}{x} \Rightarrow \frac{18}{x} - 8x + 45 = 0$$

$$\textcircled{\ast x} \quad 18 - 8x^2 + 45x = 0 \Rightarrow 8x^2 - 45x - 18 = 0$$

$$\Rightarrow (x - 6)(8x + 3) = 0$$

$$\underline{x=6}$$

known

$$\underline{x = -\frac{3}{8}}$$

$$y = \frac{9}{x} = \frac{9}{-\frac{3}{8}} = \frac{72}{-3} = -24$$

$$B\left(-\frac{3}{8}, -24\right)$$



6. (i) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^n - 1) & 5^n \end{pmatrix}$$

(6)

(ii) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(4n^2-1)$$

(6)

$$n=1 \quad \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^1 = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} \quad \text{LHS}$$

$$\begin{matrix} \text{RHS} \\ n=1 \end{matrix} \quad \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5-1) & 5^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix} \quad \begin{matrix} \therefore \text{LHS} = \text{RHS} \\ \therefore \text{true for } n=1 \end{matrix}$$

assume true for $n=k$

$$\therefore \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \quad \text{is true.}$$

$$n=k+1 \quad \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) & 5^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^k-1) - 5^k & 5^k \times 5 \end{pmatrix}$$

$$= -\frac{1}{4}(5^k-1) - \frac{4}{4}(5^k)$$

$$= -\frac{1}{4}(5^k-1+4 \times 5^k)$$

$$= -\frac{1}{4}(5 \times 5^k - 1)$$

$$= -\frac{1}{4}(5^{k+1} - 1) \quad \nearrow$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{4}(5^{k+1}-1) & 5^{k+1} \end{pmatrix}$$

\therefore true
for
 $n=k+1$

\therefore true for $n=1$, if true for $n=k$ then true for $n=k+1$

\therefore by Mathematical induction true for all $n \in \mathbb{Z}^+$ #



Question 6 continued

$$\text{ii) } n=1 \quad \sum_{r=1}^1 (2r-1)^2 = (2-1)^2 = 1^2 = 1 \quad \text{LHS}$$

$$\frac{1}{3}n(4n^2-1) = \frac{1}{3}(1)(4 \times 1^2 - 1) = \frac{1}{3}(4-1) = 1 \quad \text{RHS}$$

$\therefore \text{LHS} = \text{RHS} \quad \therefore \text{true for } n=1.$

$$n=k \quad \sum_{r=1}^k (2r-1)^2 = \frac{1}{3}k(4k^2-1) \quad \text{assume true.}$$

$$n=k+1 = \sum_{r=1}^{k+1} (2r-1)^2 = \sum_{r=1}^k (2r-1)^2 + (2(k+1)-1)^2$$

$$= \frac{1}{3}k(4k^2-1) + (2k+1)^2$$

$$= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2$$

$$= (2k+1) \left[\frac{1}{3}k(2k-1) + (2k+1) \right]$$

$$= \frac{1}{3}(2k+1) [k(2k-1) + 3(2k+1)]$$

$$= \frac{1}{3}(2k+1) [2k^2 - k + 6k + 3]$$

$$= \frac{1}{3}(2k+1) (2k^2 + 5k + 3)$$

$$= \frac{1}{3}(2k+1) (2k+3)(k+1)$$

$$= \frac{1}{3}(k+1) [(2k+1)(2k+3)]$$

$$= \frac{1}{3}(k+1) [4k^2 + 8k + 3]$$

$$= \frac{1}{3}(k+1) [4(k^2 + 2k) + 3]$$

$$= \frac{1}{3}(k+1) [4[(k+1)^2 - 1] + 3]$$

$$= \frac{1}{3}(k+1) [4(k+1)^2 - 1] \quad \therefore \text{true for } n=k+1$$

\therefore true for $n=k$, if true for $n=k$ then true for $n=k+1$

\therefore by Mathematical induction true for all $n \in \mathbb{Z}^+$



7. (i)

$$\mathbf{A} = \begin{pmatrix} 5k & 3k-1 \\ -3 & k+1 \end{pmatrix}, \text{ where } k \text{ is a real constant.}$$

Given that \mathbf{A} is a singular matrix, find the possible values of k .

(4)

(ii)

$$\mathbf{B} = \begin{pmatrix} 10 & 5 \\ -3 & 3 \end{pmatrix}$$

A triangle T is transformed onto a triangle T' by the transformation represented by the matrix \mathbf{B} .

The vertices of triangle T' have coordinates $(0, 0)$, $(-20, 6)$ and $(10c, 6c)$, where c is a positive constant.

The area of triangle T' is 135 square units.

(a) Find the matrix \mathbf{B}^{-1}

(2)

(b) Find the coordinates of the vertices of the triangle T , in terms of c where necessary.

(3)

(c) Find the value of c .

(3)

$$\text{i) Singular} \Rightarrow 5k(k+1) - (3k-1)(-3) = 0$$

$$5k^2 + 5k + 9k - 3 = 0$$

$$5k^2 + 14k - 3 = 0 \quad (5k-1)(k+3)$$

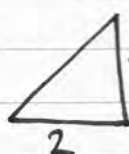
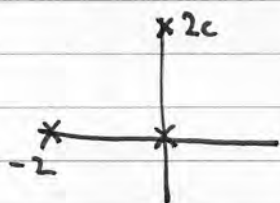
$$\therefore k = \frac{1}{5} \quad k = -3$$

$$\text{ii) } \mathbf{B}^{-1} \quad \det \mathbf{B} = 30 + 15 = 45$$

$$\mathbf{B}^{-1} = \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix}$$

$$\mathbf{T} = \frac{1}{45} \begin{pmatrix} 3 & -5 \\ 3 & 10 \end{pmatrix} \begin{pmatrix} 0 & -20 & 10c \\ 0 & 6 & 6c \end{pmatrix} = \frac{1}{45} \begin{pmatrix} 0 & -90 & 0 \\ 0 & 0 & 90c \end{pmatrix}$$

$$\mathbf{T} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2c \end{pmatrix} \quad (0,0) \quad (-2,0) \quad (0,2c)$$



$$\text{Area} = 2c = \frac{135}{45} = 3$$

$$\therefore c = \frac{3}{2}$$



8. The point $P(3p^2, 6p)$ lies on the parabola with equation $y^2 = 12x$ and the point S is the focus of this parabola.

(a) Prove that $SP = 3(1 + p^2)$ (3)

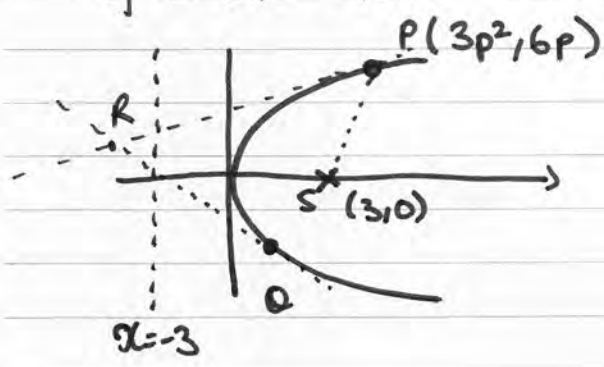
The point $Q(3q^2, 6q)$, $p \neq q$, also lies on this parabola.

The tangent to the parabola at the point P and the tangent to the parabola at the point Q meet at the point R .

(b) Find the equations of these two tangents and hence find the coordinates of the point R , giving the coordinates in their simplest form. (8)

(c) Prove that $SR^2 = SP \cdot SQ$ (3)

$y^2 = 4ax = 12x \Rightarrow \therefore a = 3 \quad \therefore \text{focus } S(3, 0)$



$SP^2 = 36p^2 + (3p^2 - 3)^2$
 $= 36p^2 + 9p^4 - 18p^2 + 9$
 $= 9p^4 + 18p^2 + 9$
 $= 9(p^4 + 2p^2 + 1)$
 $SP^2 = 9(p^2 + 1)^2$
 $\therefore SP = 3(p^2 + 1) \quad \#$

$y^2 = 12x$

$\frac{d}{dx} y^2 = \frac{d}{dx} (12x) \Rightarrow 2y \frac{dy}{dx} = 12 \quad \therefore \frac{dy}{dx} = \frac{6}{y} \Big|_{y=6p}$

$M_t \text{ at } P = \frac{1}{p}$

$y - y_1 = m(x - x_1) \Rightarrow y - 6p = \frac{1}{p}(x - 3p^2)$

$py - 6p^2 = x - 3p^2 \quad \therefore py = x + 3p^2$
 $qy = x + 3q^2$

$Q) M_t \text{ at } Q = \frac{1}{q} \quad \therefore qy = x + 3q^2$

$x = qy - 3q^2 = 3q(p+q) - 3q^2$
 $x = 3pq + 3q^2 - 3q^2$

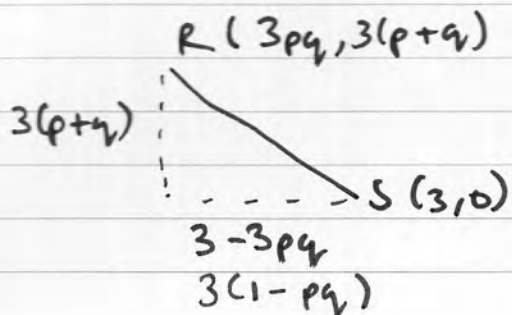
$py - qy = 3p^2 - 3q^2$
 $y(p - q) = 3(p + q)(p - q)$
 $y = 3(p + q)$

$\therefore R(3pq, 3(p+q))$



Question 8 continued

$$c) \quad SP = 3(p^2+1) \quad SQ = 3(q^2+1)$$



$$SR^2 = 3^2(p+q)^2 + 3^2(1-pq)^2$$

$$= 3^2[(p+q)^2 + (1-pq)^2]$$

$$= 3^2[p^2 + 2pq + q^2 + 1 - 2pq + p^2q^2]$$

$$= 3^2[p^2q^2 + p^2 + q^2 + 1]$$

$$= 3^2[(p^2+1)(q^2+1)]$$

$$= 3(p^2+1) \times 3(q^2+1)$$

$$\therefore SR^2 = SP \cdot SQ \quad \#$$

