

FP2 UK S14

1. The complex numbers z_1 and z_2 are given by

$$z_1 = p + 2i \text{ and } z_2 = 1 - 2i$$

where p is an integer.

(a) Find $\frac{z_1}{z_2}$ in the form $a + bi$ where a and b are real. Give your answer in its simplest form in terms of p .

(4)

Given that $\left| \frac{z_1}{z_2} \right| = 13$,

(b) find the possible values of p .

(4)

$$a) \frac{z_1}{z_2} = \frac{(p+2i) \times (1+2i)}{(1-2i) \times (1+2i)} = \frac{(p-4) + (2p+2)i}{1+4} = \frac{p-4}{5} + \frac{(2p+2)i}{5}$$

$$b) \left| \frac{z_1}{z_2} \right| = 13 \Rightarrow \frac{p^2+4}{1+4} = 169 \therefore p^2+4 = 845$$

$$\therefore p^2 = 841 \therefore p = \pm \frac{29}{1}$$

alt

$$\left| \frac{z_1}{z_2} \right| = 13 \Rightarrow \left| \left(\frac{p-4}{5} \right) + \left(\frac{2p+2}{5} \right) i \right| = 13$$

$$\Rightarrow \left(\frac{p-4}{5} \right)^2 + \left(\frac{2p+2}{5} \right)^2 = 169 = (p^2 - 8p + 16) + (4p^2 + 8p + 4) = 4225$$

$$\therefore 5p^2 + 20 = 4225$$

$$5p^2 = 4205$$

$$p^2 = 841 \therefore p = \pm \frac{29}{1}$$

2.

$$f(x) = x^3 - \frac{5}{2x^{\frac{3}{2}}} + 2x - 3, \quad x > 0$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[1.1, 1.5]$. (2)

(b) Find $f'(x)$. (2)

(c) Using $x_0 = 1.1$ as a first approximation to α , apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α , giving your answer to 3 decimal places. (3)

$$\begin{aligned} \text{a) } f(1.1) &= -1.64 & f(\alpha) &= 0 & \therefore & 1.1 < \alpha < 1.5 \\ f(1.5) &= 2.01 & & & & \text{by sign change rule.} \end{aligned}$$

$$\text{b) } f'(x) = 3x^2 + \frac{15}{4}x^{-\frac{5}{2}} + 2$$

$$\text{c) } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1.1 - \frac{(-1.64\dots)}{f'(1.1)} = 1.291$$

3. Given that 2 and $1 - 5i$ are roots of the equation

$$x^3 + px^2 + 30x + q = 0, \quad p, q \in \mathbb{R}$$

(a) write down the third root of the equation.

(1)

(b) Find the value of p and the value of q .

(5)

(c) Show the three roots of this equation on a single Argand diagram.

(2)

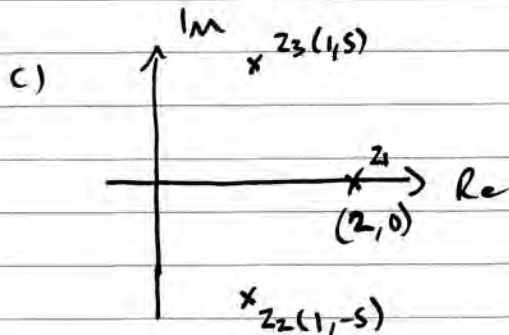
$$a) z_1 = 2 \quad z_2 = 1 - 5i \quad z_3 = z_2^* = \underline{1 + 5i}$$

$$b) (x-2)[(x-(1-5i))(x-(1+5i))] = 0$$

$$\Rightarrow (x-2)(x^2 - 2x + 26) = 0$$

$$\Rightarrow \begin{array}{r} \textcircled{x} \textcircled{x} \quad x^3 - 2x^2 + 26x \\ \textcircled{x-2} \quad -2x^2 + 4x - 52 \quad + \\ \hline \end{array}$$

$$\begin{array}{r} x^3 - 4x^2 + 30x - 52 = 0 \quad \therefore p = -4 \\ \checkmark \quad \quad \quad \checkmark \quad \quad \quad \checkmark \quad \quad \quad \checkmark \quad \quad \quad \checkmark \\ q = -52 \\ \underline{2} \end{array}$$



4. (i) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 3 & 1 \end{pmatrix},$$

(a) find \mathbf{AB} .

(b) Explain why $\mathbf{AB} \neq \mathbf{BA}$.

(4)

(ii) Given that

$$\mathbf{C} = \begin{pmatrix} 2k & -2 \\ 3 & k \end{pmatrix}, \text{ where } k \text{ is a real number}$$

find \mathbf{C}^{-1} , giving your answer in terms of k .

(3)

$$a) \mathbf{AB} = \begin{pmatrix} 4 & 5 & 6 \\ 5 & -6 & 11 \\ 13 & 11 & 21 \end{pmatrix}$$

b) matrix multiplication is not commutative.

$\mathbf{AB} = 3 \times 3$ matrix $\mathbf{BA} = 3 \times 2$ matrix

$$ii) \det \mathbf{C} = 2k^2 + 6$$

$$\therefore \mathbf{C}^{-1} = \frac{1}{2k^2 + 6} \begin{pmatrix} k & 2 \\ -3 & 2k \end{pmatrix}$$

5. (a) Use the standard results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$ to show that

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(4n^2-1) \quad (6)$$

- (b) Hence show that

$$\sum_{r=2n+1}^{4n} (2r-1)^2 = an(bn^2-1)$$

where a and b are constants to be found.

(3)

$$a \sum_1^n (2r-1)^2 = 4 \sum_1^n r^2 - 4 \sum_1^n r + \sum_1^n 1$$

$$= 4 \times \frac{1}{6}n(n+1)(2n+1) - 4 \times \frac{1}{2}n(n+1) + n$$

$$= \frac{1}{3}n [2(n+1)(2n+1) - 6(n+1) + 3]$$

$$= \frac{1}{3}n [4n^2 + 6n + 2 - 6n - 6 + 3]$$

$$= \frac{1}{3}n [4n^2 - 1] \neq$$

$$b) \sum_{2n+1}^{4n} = \sum_1^{4n} - \sum_1^{2n} = \frac{1}{3}(4n)[4(4n)^2-1] - \frac{1}{3}(2n)[4(2n)^2-1]$$

$$= \frac{4}{3}n(64n^2-1) - \frac{2}{3}n(16n^2-1)$$

$$= \frac{2}{3}n [2(64n^2-1) - (16n^2-1)]$$

$$= \frac{2}{3}n [112n^2-1]$$

6. The rectangular hyperbola H has cartesian equation $xy = c^2$.

The point $P\left(ct, \frac{c}{t}\right)$, $t > 0$, is a general point on H .

(a) Show that an equation of the tangent to H at the point P is

$$t^2y + x = 2ct \quad (4)$$

An equation of the normal to H at the point P is $t^3x - ty = ct^4 - c$

Given that the normal to H at P meets the x -axis at the point A and the tangent to H at P meets the x -axis at the point B ,

(b) find, in terms of c and t , the coordinates of A and the coordinates of B . (2)

Given that $c = 4$,

(c) find, in terms of t , the area of the triangle APB . Give your answer in its simplest form. (3)

$$a) y = c^2x^{-1} \quad \frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

$$P\left(ct, \frac{c}{t}\right) \quad M_t = \frac{-c^2}{t^2t^2} = -\frac{1}{t^2}$$

$$\therefore y - y_1 = m(x - x_1) \Rightarrow y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

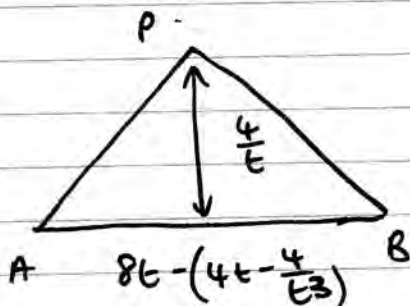
$$\Rightarrow yt^2 - ct = -x + ct \quad \therefore t^2y + x = 2ct$$

$$b) y = 0 \Rightarrow t^3x = ct^4 - c$$

$$x = ct - \frac{c}{t^3} \Rightarrow A\left(ct - \frac{c}{t^3}, 0\right)$$

$$0 + x = 2ct \Rightarrow B(2ct, 0)$$

$$c) c = 4 \quad A\left(4t - \frac{4}{t^3}, 0\right) \quad B(8t, 0) \quad P\left(4t, \frac{4}{t}\right)$$



$$\text{Area} = \left(2t + \frac{2}{t^3}\right) \left(\frac{4}{t}\right)$$

$$= 8 + \frac{8}{t^4}$$

2

$$= 4t + \frac{4}{t^3}$$

7. (i) In each of the following cases, find a 2×2 matrix that represents

(a) a reflection in the line $y = -x$,

(b) a rotation of 135° anticlockwise about $(0, 0)$,

(c) a reflection in the line $y = -x$ followed by a rotation of 135° anticlockwise about $(0, 0)$. (4)

(ii) The triangle T has vertices at the points $(1, k)$, $(3, 0)$ and $(11, 0)$, where k is a constant.

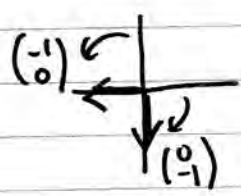
Triangle T is transformed onto the triangle T' by the matrix

$$\begin{pmatrix} 6 & -2 \\ 1 & 2 \end{pmatrix}$$

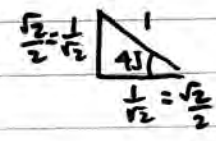
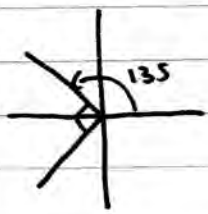
Given that the area of triangle T' is 364 square units, find the value of k .

(6)

a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

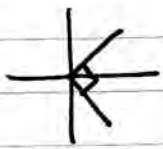
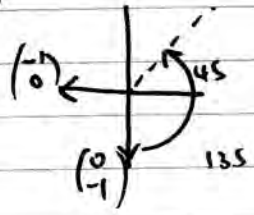


b)



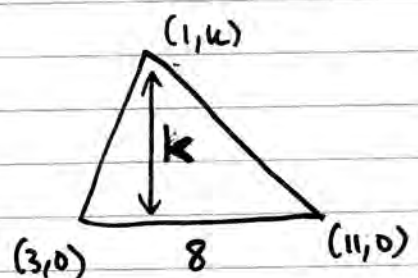
$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

c)



$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \text{ alt } \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

ii) $\det(M) = 12 - (-2) = 14 \quad \therefore \text{area of } T = \frac{364}{14} = 26$



$$\text{Area} = \frac{1}{2}(8) \times k = 4k = 26$$
$$\therefore k = 6.5$$

8. The points $P(4k^2, 8k)$ and $Q(k^2, 4k)$, where k is a constant, lie on the parabola C with equation $y^2 = 16x$.

The straight line l_1 passes through the points P and Q .

- (a) Show that an equation of the line l_1 is given by

$$3ky - 4x = 8k^2 \quad (4)$$

The line l_2 is perpendicular to the line l_1 and passes through the focus of the parabola C . The line l_2 meets the directrix of C at the point R .

- (b) Find, in terms of k , the y coordinate of the point R . (7)

$$a) M_{PQ} = \frac{4k}{3k^2} = \frac{4}{3k} \quad (k^2, 4k)$$

$$y - y_1 = m(x - x_1) \quad y - 4k = \frac{4}{3k}(x - k^2)$$

$$3ky - 12k^2 = 4x - 4k^2$$

$$\therefore 3ky - 4x = 8k^2 \quad \#$$

$$b) M_{l_2} = -\frac{3k}{4} \quad y^2 = 4ax = 16x \quad \therefore a = 4$$

$$\therefore \text{focus } (a, 0) = (4, 0)$$

$$y = -\frac{3k}{4}(x - 4) \Rightarrow 4y = 12k - 3kx$$

$$\text{directrix } x = -a \quad \Rightarrow 4y = 12k + 3ka$$

$$\therefore 4y = 12k + 12k$$

$$4y = 24k$$

$$y = 6k$$

9. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 8^n - 2^n$$

is divisible by 6

(6)

$$n=1 \quad f(1) = 8^1 - 2^1 = 8 - 2 = 6 = 6 \times 1 \quad \therefore \text{true if } n=1$$

$$n=2 \quad f(2) = 8^2 - 2^2 = 64 - 4 = 60 = 6 \times 10 \quad \therefore \text{true for } n=2$$

assume true for $n=k$ $\therefore f(k) = 8^k - 2^k$ is divisible by 6

$$\begin{aligned} n=k+1 \quad f(k+1) &= 8^{k+1} - 2^{k+1} \\ &= 8(8^k) - 2(2^k) \\ &= 6(8^k) + 2(8^k) - 2(2^k) \\ &= 6(8^k) + 2[8^k - 2^k] \\ &= 6(8^k) + 2f(k) \quad \therefore \text{divisible by 6.} \end{aligned}$$

\therefore true for $n=1$, true for $n=k+1$ if true for $n=k$

\therefore by Mathematical induction true for $n \in \mathbb{Z}^+$