

FP2 S14 (R)

1. The roots of the equation

$$2z^3 - 3z^2 + 8z + 5 = 0$$

are z_1, z_2 and z_3

Given that $z_1 = 1 + 2i$, find z_2 and z_3

(5)

$$z_2 = 1 - 2i$$

$$(2z - \alpha)(z - (1 + 2i))(z - (1 - 2i)) = 0$$

$$(2z - \alpha)(z^2 - 2z + 5) = 0 \quad -5\alpha = 5 \quad \therefore \alpha = -1$$

$$(2z + 1) = 0 \quad \therefore z_3 = -\frac{1}{2}$$

$$z_1 = \frac{1+2i}{2} \quad z_2 = \frac{1-2i}{2} \quad z_3 = \frac{-1}{2}$$

2.

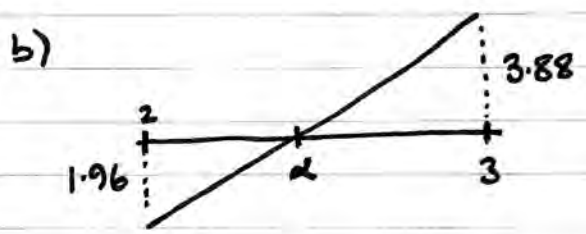
$$f(x) = 3 \cos 2x + x - 2, \quad -\pi \leq x < \pi$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval $[2, 3]$. (2)

(b) Use linear interpolation once on the interval $[2, 3]$ to find an approximation to α .
Give your answer to 3 decimal places. (3)

(c) The equation $f(x) = 0$ has another root β in the interval $[-1, 0]$. Starting with this interval, use interval bisection to find an interval of width 0.25 which contains β . (4)

a) $f(2) = -1.96$ $f(x) = 0$ so by sign change rule
 $f(3) = 3.88$ $2 < \alpha < 3$

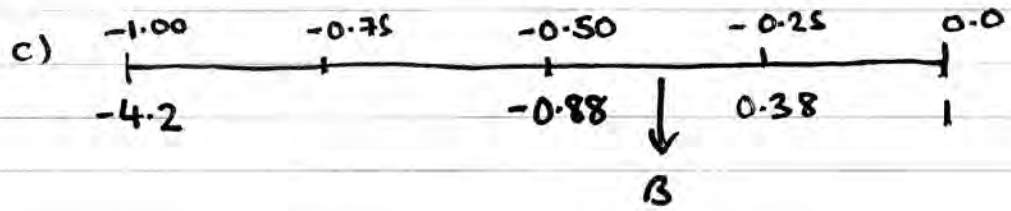


$$\frac{\alpha - 2}{1.96} = \frac{3 - \alpha}{3.88}$$

$$3.88\alpha - 7.761.. = 5.882.. - 1.96\alpha$$

$$\therefore 5.84144.. \alpha = 13.6438..$$

$$\therefore \alpha = \underline{2.336}$$



$$\therefore -0.5 < \beta < -0.25 \quad \beta \in (-0.5; 0.25)$$

3. (i)

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

(a) Describe fully the single transformation represented by the matrix **A**. (2)

The matrix **B** represents an enlargement, scale factor -2 , with centre the origin.

(b) Write down the matrix **B**. (1)

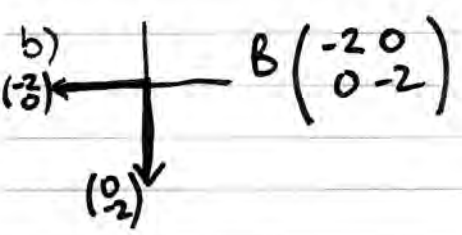
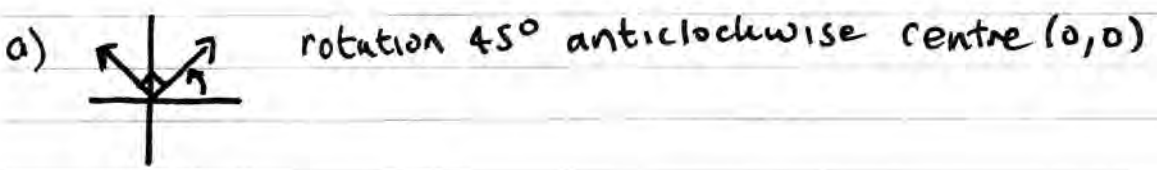
(ii)

$$M = \begin{pmatrix} 3 & k \\ -2 & 3 \end{pmatrix}, \text{ where } k \text{ is a positive constant.}$$

Triangle *T* has an area of 16 square units.

Triangle *T* is transformed onto the triangle *T'* by the transformation represented by the matrix **M**.

Given that the area of the triangle *T'* is 224 square units, find the value of *k*. (3)



ii) $\det M = 9 + 2k$

$T \rightarrow T' = \text{enlargement sf} = 14$

$\therefore 9 + 2k = 14 \Rightarrow 2k = 5$

$\therefore k = \frac{2.5}{2}$

4. The complex number z is given by

$$z = \frac{p + 2i}{3 + pi}$$

where p is an integer.

(a) Express z in the form $a + bi$ where a and b are real. Give your answer in its simplest form in terms of p .

(4)

(b) Given that $\arg(z) = \theta$, where $\tan \theta = 1$ find the possible values of p .

(5)

$$a) \quad z = \frac{p + 2i}{3 + pi} \times \frac{(3 - pi)}{(3 - pi)} = \frac{5p + (6 - p^2)i}{9 + p^2}$$

$$z = \frac{5p}{9 + p^2} + \frac{6 - p^2}{9 + p^2}i$$

$$b) \quad \theta = \tan^{-1}(1) = \frac{\pi}{4}, \frac{5\pi}{4}$$



$$\frac{6 - p^2}{9 + p^2}$$

$$\therefore \frac{5p}{9 + p^2} = \frac{6 - p^2}{9 + p^2}$$

$$\therefore 5p = 6 - p^2$$

$$\therefore p^2 + 5p - 6 = 0$$

$$\therefore (p + 6)(p - 1) = 0$$

$$p = -6 \quad p = 1$$

2 7

5. (a) Use the standard results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^3$ to show that

$$\sum_{r=1}^n r(r^2 - 3) = \frac{1}{4}n(n+1)(n+3)(n-2)$$

(5)

- (b) Calculate the value of $\sum_{r=10}^{50} r(r^2 - 3)$

(3)

$$\begin{aligned} \sum_1^n r(r^2-3) &= \sum_1^n r^3 - 3\sum_1^n r = \frac{1}{4}n^2(n+1)^2 - 3\frac{1}{2}n(n+1) \\ &= \frac{1}{4}n(n+1)[n(n+1) - 6] \\ &= \frac{1}{4}n(n+1)(n^2+n-6) \\ &= \frac{1}{4}n(n+1)(n+3)(n-2) \quad \# \end{aligned}$$

$$\begin{aligned} \text{b) } \sum_{10}^{50} &= \sum_1^{50} - \sum_1^9 \Rightarrow \frac{1}{4}(50)(51)(53)(48) - \frac{1}{4}(9)(10)(12)(7) \\ &= 1619910 \end{aligned}$$

6.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

Given that $\mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B})$,

(a) calculate the matrix \mathbf{M} ,

(6)

(b) find the matrix \mathbf{C} such that $\mathbf{MC} = \mathbf{A}$.

(4)

$$a) \mathbf{M} = (\mathbf{A} + \mathbf{B})(2\mathbf{A} - \mathbf{B}) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -7 & -2 \end{pmatrix}$$

$$b) \mathbf{MC} = \mathbf{A} \Rightarrow \mathbf{M}^{-1}\mathbf{MC} = \mathbf{M}^{-1}\mathbf{A} \Rightarrow \mathbf{C} = \mathbf{M}^{-1}\mathbf{A}$$

$$\det(\mathbf{M}) = \begin{vmatrix} 1 & -1 \\ -7 & -2 \end{vmatrix} = -2 - 7 = -9$$

~~$$\mathbf{M}^{-1} = \frac{1}{-9} \begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} ; \mathbf{C} = \frac{1}{-9} \begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$~~

$$\mathbf{M}^{-1} = -\frac{1}{9} \begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix}$$

~~$$\mathbf{C} = \frac{1}{-9} \begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -5 & -2 \\ 13 & 7 \end{pmatrix}$$~~

$$\mathbf{C} = \frac{1}{9} \begin{pmatrix} -2 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -5 & -2 \\ 13 & 7 \end{pmatrix}$$

7. The parabola C has cartesian equation $y^2 = 4ax$, $a > 0$

The points $P(ap^2, 2ap)$ and $P'(ap^2, -2ap)$ lie on C .

(a) Show that an equation of the normal to C at the point P is

$$y + px = 2ap + ap^3 \tag{5}$$

(b) Write down an equation of the normal to C at the point P' . (1)

The normal to C at P meets the normal to C at P' at the point Q .

(c) Find, in terms of a and p , the coordinates of Q . (2)

Given that S is the focus of the parabola,

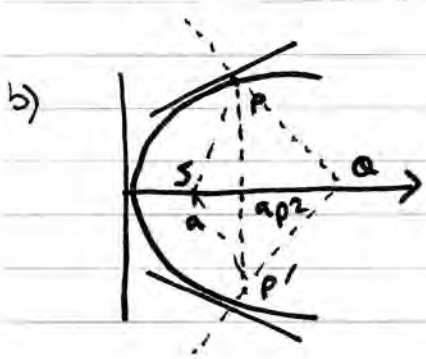
(d) find the area of the quadrilateral $SPQP'$. (3)

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax) \Rightarrow 2y \frac{dy}{dx} = 4a \therefore \frac{dy}{dx} = \frac{4a}{2y}$$

$$\text{at } P(ap^2, 2ap) \quad \frac{dy}{dx} = \frac{4a}{4ap} = \frac{1}{p} \therefore m_n = -p$$

$$y - 2ap = -p(x - ap^2) \Rightarrow y - 2ap = -px + ap^3$$

$$\therefore y + px = 2ap + ap^3 \quad \#$$



$$\text{at } P', m_n = \left(\frac{-4ap}{4a}\right) = p \quad y + 2ap = p(x - ap^2)$$

$$\therefore y - px = -2ap - ap^3$$

c) Normals meet at x -axis $\Rightarrow y=0 \therefore px = 2ap + ap^3$
 $x = 2a + ap^2$

d) $S(a, 0) \quad Q(2a + ap^2, 0)$

$$\therefore \text{Area} = \frac{1}{2} \vec{SQ} \times \vec{P'P}$$

$$= \frac{1}{2} (2a + ap^2 - a)(4ap) = 2ap(a + ap^2) = 2a^2p(1 + p^2)$$

8. The rectangular hyperbola H has equation $xy = c^2$, where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$, $t \neq 0$, is a general point on H .

An equation for the tangent to H at P is given by

$$y = -\frac{1}{t^2}x + \frac{2c}{t}$$

The points A and B lie on H .

The tangent to H at A and the tangent to H at B meet at the point $\left(-\frac{6}{7}c, \frac{12}{7}c\right)$.

Find, in terms of c , the coordinates of A and the coordinates of B .

(5)

$$\frac{12c}{7} = -\frac{-6c}{7t^2} + \frac{2c}{t} = \frac{6c}{7t^2} + \frac{14ct}{7t^2}$$

$$\therefore \frac{12ct^2}{7t^2} = \frac{6c + 14ct}{7t^2} \quad \therefore 12ct^2 = 6c + 14ct$$

$$\Rightarrow 6t^2 - 7t - 6 = 0$$

$$(3t + 1)(2t - 3) = 0$$

$$\therefore t = -\frac{1}{3} \quad t = \frac{3}{2}$$

$$t = -\frac{1}{3} \quad A\left(-\frac{1}{3}c, -3c\right)$$

$$t = \frac{3}{2} \quad B\left(\frac{3}{2}c, \frac{2c}{3}\right)$$

9. (a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^n (r+1)2^{r-1} = n2^n$$

(5)

(b) A sequence of numbers is defined by

$$u_1 = 0, \quad u_2 = 32,$$

$$u_{n+2} = 6u_{n+1} - 8u_n \quad n \geq 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 4^{n+1} - 2^{n+3}$$

(7)

$$a) \quad n=1 \quad \sum_1^1 (r+1)2^{r-1} = (1+1)2^{1-1} = 2 \times 2^0 = 2$$

$$n2^n = 1(2^1) = 2 \quad \therefore \text{true for } n=1$$

$$\text{assume true for } n=k \quad \therefore \sum_1^k (r+1)2^{r-1} = k2^k$$

$$n=k+1 \quad \sum_1^{k+1} (r+1)2^{r-1}$$

$$= (k+1+1)2^{k+1-1} + \sum_1^k (r+1)2^{r-1}$$

$$= (k+2)2^k + k2^k = (2k+2)2^k = 2(k+1)2^k$$

$$= (k+1) \times 2 \times 2^k$$

$$= (k+1)2^{k+1}$$

$$n=k+1 \quad n2^n = (k+1)2^{k+1}$$

\therefore true for $n=1$, if true for $n=k$, true for $n=k+1$

\therefore by Mathematical Induction true for $n \in \mathbb{Z}^+$

$$b) \quad n=1 \quad u_1=0 \quad u_1 = 4^{1+1} - 2^{1+3} = 4^2 - 2^4 = 16 - 16 = 0$$

\therefore true for $n=1$

$$n=2 \quad u_2=32 \quad u_2 = 4^{2+1} - 2^{2+3} = 4^3 - 2^5 = 64 - 32 = 32 \quad \therefore \text{true for } n=2$$

assume true for $n=k$ and $n=k+1$

$$\therefore u_k = 4^{k+1} - 2^{k+3}$$

$$u_{k+1} = 4^{k+2} - 2^{k+4}$$

$n=k+2$

$$u_{k+2} = 4^{k+3} - 2^{k+5}$$

$$u_{k+2} = 6u_{k+1} - 8u_k$$

$$= 6(4^{k+2} - 2^{k+4}) - 8(4^{k+1} - 2^{k+3})$$

$$= 6 \times 4^{k+2} - 6 \times 2^{k+4} - 8 \times 4^{k+1} + 8 \times 2^{k+3}$$

$$= 6 \times 4^{k+2} - 3 \times 2^{k+5} - 2 \times 4^{k+2} + 2 \times 2^{k+5}$$

$$= 4 \times 4^{k+2} - 1 \times 2^{k+5}$$

$$= 4^{k+3} - 2^{k+5} \quad \#$$

\therefore true for $n=1, n=2$

if true for $n=k, n=k+1$ then true for $n=k+2$

\therefore by Mathematical Induction true for $n \in \mathbb{Z}^+$