FP1 June 2013 (un)

1.

$$\mathbf{M} = \begin{pmatrix} x & x-2 \\ 3x-6 & 4x-11 \end{pmatrix}$$

Given that the matrix **M** is singular, find the possible values of x.

(4)

Singular $\Rightarrow$ det $= ad - bc = 0$

$\Rightarrow x(4x-11) - (x-2)(3x-6) = 4x^2 - 11x - 3x^2 + 12x - 12 = 0$

$= (x^2 + x - 12) = 0 \Rightarrow (x+4)(x-3) = 0 \Rightarrow \underline{x=3}, \underline{x=-4}$

2. $$f(x) = \cos(x^2) - x + 3, \qquad 0 < x < \pi$$

(a) Show that the equation $f(x) = 0$ has a root α in the interval [2.5, 3].

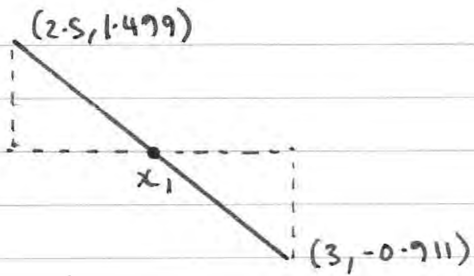
(2)

(b) Use linear interpolation once on the interval [2.5, 3] to find an approximation for α, giving your answer to 2 decimal places.

(3)

a) $f(2.5) = 1.499 > 0$ $\therefore$ by sign change law
$f(3) = -0.911 < 0$ $\alpha \in [2.5, 3]$

b)



$$\frac{3 - x}{x - 2.5} = \frac{0.911}{1.499}$$

$4.497 - 1.499x = 0.911x - 2.775$

$\Rightarrow 6.7745 = 2.41x$

$\therefore x = \underline{2.81}$ (2dp)

3. Given that $x = \frac{1}{2}$ is a root of the equation

$$2x^3 - 9x^2 + kx - 13 = 0, \qquad k \in \mathbb{R}$$

find

(a) the value of k, **(3)**

(b) the other 2 roots of the equation. **(4)**

a) $f(\frac{1}{2}) = \frac{1}{4} - \frac{9}{4} + \frac{1}{2}k - 13 = 0 \Rightarrow \frac{1}{2}k = 15 \quad \therefore \underline{k = 30}$

b) $(x - \frac{1}{2})(2x^2 + Ax + 26) = 0 \Rightarrow Ax^2 - 1x^2 \equiv -9x^2 \therefore A = -8$

$\Rightarrow 2x^2 - 8x + 26 = 0 \Rightarrow x^2 - 4x = -13$

$\Rightarrow (x-2)^2 - 4 = -13 \Rightarrow (x-2)^2 = -9 \Rightarrow x - 2 = \pm 3i$

$\therefore x = 2 + 3i, \ x = 2 - 3i$

4. The rectangular hyperbola H has Cartesian equation $xy = 4$

The point $P\left(2t, \frac{2}{t}\right)$ lies on H, where $t \neq 0$

(a) Show that an equation of the normal to H at the point P is

$$ty - t^3x = 2 - 2t^4$$

(5)

The normal to H at the point where $t = -\frac{1}{2}$ meets H again at the point Q.

(b) Find the coordinates of the point Q.

(4)

a) $y = 4x^{-1} \Rightarrow \frac{dy}{dx} = -4x^{-2} = \frac{-4}{x^2}$

$M_t\Big|_{x=2t} = \frac{-4}{(2t)^2} = \frac{-4}{4t^2} = \frac{-1}{t^2} \Rightarrow m_n = t^2$

at $\left(2t, \frac{2}{t}\right)$ $\quad y - \frac{2}{t} = t^2(x - 2t) \Rightarrow ty - 2 = t^3(x - 2t)$

$\Rightarrow ty - 2 = t^3x - 2t^4 \quad \Rightarrow ty - t^3x = 2 - 2t^4$ #

b) $t = -\frac{1}{2} \Rightarrow -\frac{1}{2}y + \frac{1}{8}x = 2 - \frac{1}{8}$ $(\times 8)$

$-4y + x = 16 - 1 \Rightarrow x - 4y = 15 \qquad x = \frac{4}{y}$

$\Rightarrow \frac{4}{y} - 4y = 15 \Rightarrow 4 - 4y^2 = 15y \Rightarrow 4y^2 + 15y - 4 = 0$

$\Rightarrow (4y - 1)(y + 4) = 0 \qquad y = +\frac{1}{4}, \ y = -4$

$\therefore \left(16, \frac{1}{4}\right) \qquad (-1, -4)$

5. (a) Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$ to show that

$$\sum_{r=1}^{n} (r+2)(r+3) = \frac{1}{3}n(n^2 + 9n + 26)$$

for all positive integers n.

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n} (r+2)(r+3) = \frac{2}{3}n(an^2 + bn + c)$$

where a, b and c are integers to be found.

(4)

$= \sum r^2 + 5\sum r + 6\sum 1 = \frac{1}{6}n(n+1)(2n+1) + \frac{5n}{2}(n+1) + 6n$

$= \frac{1}{6}n\left[(n+1)(2n+1) + 15(n+1) + 36\right] = \frac{1}{6}n\left[2n^2+3n+1+15n+15+36\right]$

$= \frac{1}{6}n\left[2n^2+18n+52\right] = \frac{1}{3}n\left[n^2+9n+26\right]$ #

b) $\sum_{r=n+1}^{3n} = \sum_{r=1}^{3n} - \sum_{r=1}^{n} = \frac{1}{3}(3n)\left[(3n)^2+9(3n)+26\right] - \frac{1}{3}n\left[n^2+9n+26\right]$

$= \frac{1}{3}n\left[27n^2+81n+78-n^2-9n-26\right] = \frac{1}{3}n\left[26n^2+72n+52\right]$

$\therefore \frac{2}{3}n\left[13n^2+36n+26\right]$.

6. A parabola C has equation $y^2 = 4ax, \quad a > 0$

The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on C, where $p \neq 0, q \neq 0, p \neq q$.

(a) Show that an equation of the tangent to the parabola at P is

$$py - x = ap^2$$

(4)

(b) Write down the equation of the tangent at Q.

(1)

The tangent at P meets the tangent at Q at the point R.

(c) Find, in terms of p and q, the coordinates of R, giving your answers in their simplest form.

(4)

Given that R lies on the directrix of C,

(d) find the value of pq.

(2)

a) $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \quad \therefore m_t\Big|_{y=2ap} = \frac{2a}{2ap} = \frac{1}{p}$

at $(ap^2, 2ap) \Rightarrow y - 2ap = \frac{1}{p}(x - ap^2) \Rightarrow py - 2ap^2 = x - ap^2$

$\therefore py - x = ap^2$ # $\qquad$ b) $qy - x = aq^2$

c) $x = py - ap^2, \quad x = qy - aq^2$

$\Rightarrow py - ap^2 = qy - aq^2 \Rightarrow py - qy = ap^2 - aq^2$

$\Rightarrow (p-q)y = a(p^2 - q^2) \Rightarrow \cancel{(p-q)}y = a(p+q)\cancel{(p-q)}$

$\therefore y = a(p+q) \qquad x = p[a(p+q)] - ap^2$

$\Rightarrow x = ap^2 + apq - ap^2 \Rightarrow x = apq \quad y = a(p+q)$

d) along directrix $x = -a \Rightarrow -a = apq \quad \therefore \underline{pq = -1}$

7. $$z_1 = 2 + 3\text{i}, \quad z_2 = 3 + 2\text{i}, \quad z_3 = a + b\text{i}, \quad a, b \in \mathbb{R}$$

(a) Find the exact value of $|z_1 + z_2|$. **(2)**

Given that $w = \dfrac{z_1 z_3}{z_2}$,

(b) find w in terms of a and b, giving your answer in the form $x + \text{i}y, \quad x, y \in \mathbb{R}$ **(4)**

Given also that $w = \dfrac{17}{13} - \dfrac{7}{13}\text{i}$,

(c) find the value of a and the value of b, **(3)**

(d) find arg w, giving your answer in radians to 3 decimal places. **(2)**

a) $|2+3i+3+2i| = |5+5i| = \sqrt{5^2+5^2} = 5\sqrt{2}$

b) $w = \dfrac{(2+3i)(a+bi)}{(3+2i)} \times \dfrac{(3-2i)}{(3-2i)} = \dfrac{(12+5i)(a+bi)}{13}$

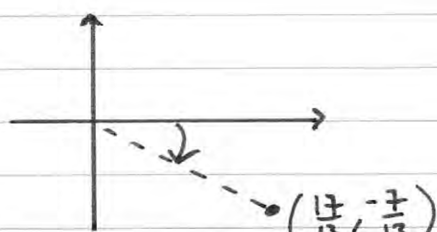
$w = \dfrac{12a-5b}{13} + \dfrac{(5a+12b)i}{13}$

c) $12a - 5b = 17 \quad (\times 12) \qquad 144a - 60b = 204$

$5a + 12b = -7 \quad (\times 5) \qquad 25a + 60b = -35$

$169a = 169 \qquad \therefore a = 1, b = -1$

d)



$\arg w = -\tan\left(\dfrac{\frac{7}{13}}{\frac{17}{13}}\right)$

$\therefore \arg w = -0.391^c$

8.

$$\mathbf{A} = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

and $\mathbf{I}$ is the 2×2 identity matrix.

(a) Prove that

$$\mathbf{A}^2 = 7\mathbf{A} + 2\mathbf{I}$$

(2)

(b) Hence show that

$$\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A} - 7\mathbf{I})$$

(2)

The transformation represented by $\mathbf{A}$ maps the point P onto the point Q.

Given that Q has coordinates $(2k + 8, -2k - 5)$, where k is a constant,

(c) find, in terms of k, the coordinates of P.

(4)

a) $A^2 = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix}$

$7A + 2I = \begin{pmatrix} 42 & -14 \\ -28 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix} \Rightarrow A^2 = 7A + 2I$

b) $A^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}$

$A - 7I = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -4 & -6 \end{pmatrix}$

$\therefore A^{-1} = \frac{1}{2}(A - 7I)$ #

alt

$AA = 7A + 2I$

$\Rightarrow A^{-1}AA = 7A^{-1}A + 2A^{-1}I$

$\Rightarrow A = 7I + 2A^{-1}$

$\Rightarrow 2A^{-1} = A - 7I$

$\Rightarrow A^{-1} = \frac{1}{2}(A - 7I)$ #

c) $AP = Q \Rightarrow \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2k+8 \\ -2k-5 \end{pmatrix} \Rightarrow \begin{matrix} 6x - 2y = 2k+8 \\ -4x + y = -2k-5 \end{matrix}$

$\Rightarrow$ $6x - 2y = 2k + 8$ +

$-8x + 2y = -4k - 10$

$-2x = -2k - 2$

$\therefore x = k + 1$

$y = -2k - 5 + 4(k+1)$

$y = -2k - 5 + 4k + 4$

$y = 2k - 1$

blank

9. (a) A sequence of numbers is defined by

$$u_1 = 8$$

$$u_{n+1} = 4u_n - 9n, \quad n \geqslant 1$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = 4^n + 3n + 1$$

(5)

(b) Prove by induction that, for $m \in \mathbb{Z}^+$,

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^m = \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix}$$

(5)

$u_1 = 8$ $\qquad$ $u_1 = 4^1 + 3(1) + 1 = 8$ ✓

$u_2 = 4(8) - 9 \times 1 = 23$ $\qquad$ $u_2 = 4^2 + (3)(2) + 1 = 23$ ✓

assume true for $n = k$ $\quad \therefore\ u_k = 4^k + 3k + 1$

$\therefore\ u_{k+1} = 4^{k+1} + 3(k+1) + 1 = 4^{k+1} + 3k + 4$

$u_{k+1} = 4u_k - 9k = 4(4^k + 3k + 1) - 9k$
$= 4^{k+1} + 12k + 4 - 9k = 4^{k+1} + 3k + 4$ #

$\therefore$ true for $n=1, n=2$; true for $n = k+1$ if true for $n = k$
$\therefore$ by induction, true for all $n \in \mathbb{Z}^+$

b) $m = 1$ $\quad \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ $\qquad \begin{pmatrix} 2m+1 & -4m \\ m & 1-2m \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$ ✓

assume true for $n = k$ $\quad \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix}$

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^k = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2k+1 & -4k \\ k & 1-2k \end{pmatrix} = \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2(k+1)+1 & -4(k+1) \\ (k+1) & 1-2(k+1) \end{pmatrix} = \begin{pmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{pmatrix}$$

$\therefore$ true for $m=1$, true for $m = k+1$ if true for $m = k$
$\therefore$ by induction, true for all $m \in \mathbb{Z}^+$