FP1 (int) June 13

1. The complex numbers z and w are given by

$$z = 8 + 3\text{i}, \quad w = -2\text{i}$$

Express in the form $a + b\text{i}$, where a and b are real constants,

(a) $z - w$,

(1)

(b) zw.

(2)

a) $8 + 5i$

b) $(8+3i)(-2i) = -16i - 6i^2 = 6 - 16i$

2. (i) $$\mathbf{A} = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix}, \text{ where } k \text{ is a constant}$$

Given that

$$\mathbf{B} = \mathbf{A} + 3\mathbf{I}$$

where **I** is the 2×2 identity matrix, find

(a) **B** in terms of k,

(2)

(b) the value of k for which **B** is singular.

(2)

(ii) Given that

$$\mathbf{C} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \quad \mathbf{D} = (2 \ \ -1 \ \ 5)$$

and

$$\mathbf{E} = \mathbf{CD}$$

find **E**.

(2)

a) $A + 3I = \begin{pmatrix} 2k+1 & k \\ -3 & -5 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2k+4 & k \\ -3 & -2 \end{pmatrix}$

b) Singular $\Rightarrow$ det $=$ ad $-$ bc $= 0$

$(2k+4)(-2) - (-3)(k) = 0 \Rightarrow -4k - 8 + 3k = 0$

$\Rightarrow -k = 8 \Rightarrow \underline{k = -8}$

ii) $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} (2, -1, 5) = \begin{pmatrix} 4 & -2 & 10 \\ -6 & 3 & -15 \\ 8 & -4 & 20 \end{pmatrix}$

3. $$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

(a) Show that the equation $f(x) = 0$ has a root α between $x = 2$ and $x = 2.5$

(2)

(b) Starting with the interval $[2, 2.5]$ use interval bisection twice to find an interval of width 0.125 which contains α.

(3)

The equation $f(x) = 0$ has a root β in the interval $[-2, -1]$.

(c) Taking -1.5 as a first approximation to β, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to β.
Give your answer to 2 decimal places.

(5)

a) $f(2) = -1$ $\quad\therefore$ by sign change rule root $\alpha \in [2, 2.5]$
$f(2.5) = 3.41$

b)

α

x: 2.000 (2.125 2.250) 2.500

$f(x)$: -1 $\quad -0.28$ $\quad 0.67$ $\quad 3.4$

$\therefore \quad \alpha \in (2.125, 2.250) \qquad 2.125 < \alpha < 2.25$

c) $f'(x) = 2x^3 - 3x^2 + 1 \qquad x_1 = -1.5$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.5 - \frac{\frac{45}{32}}{-\frac{25}{2}} = -1.39 \text{ (2dp)}$$

$\beta = -1.39$

4. $$f(x) = (4x^2 + 9)(x^2 - 2x + 5)$$

(a) Find the four roots of $f(x) = 0$

(4)

(b) Show the four roots of $f(x) = 0$ on a single Argand diagram.

(2)

a) $4x^2+9=0 \Rightarrow x^2 = -\frac{9}{4} \Rightarrow x = \pm\frac{3}{2}i$

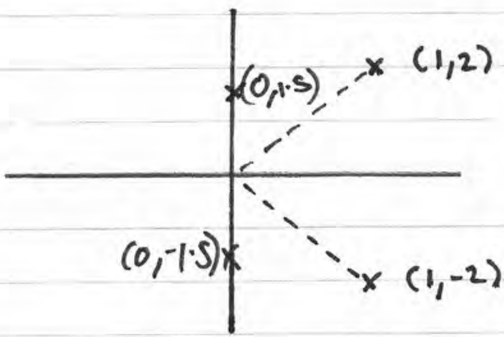
$x^2-2x+5=0 \Rightarrow (x-1)^2-1 = -5$

$\Rightarrow (x-1)^2 = -4$

$\Rightarrow x-1 = \pm 2i$

$\Rightarrow x = 1 \pm 2i$

$\therefore x = \frac{3}{2}, -\frac{3}{2}, 1+2i, 1-2i$



5.

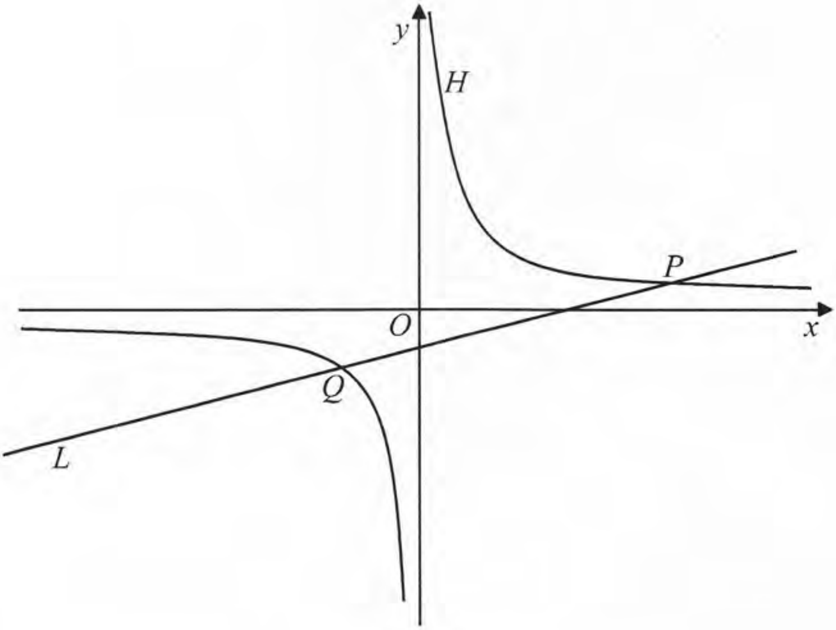


Figure 1

Figure 1 shows a rectangular hyperbola H with parametric equations

$$x = 3t, \quad y = \frac{3}{t}, \quad t \neq 0$$

The line L with equation $6y = 4x - 15$ intersects H at the point P and at the point Q as shown in Figure 1.

(a) Show that L intersects H where $4t^2 - 5t - 6 = 0$

(3)

(b) Hence, or otherwise, find the coordinates of points P and Q.

(5)

a) $6\left(\frac{3}{t}\right) = 4(3t) - 15 \Rightarrow \frac{18}{t} = 12t - 15 \quad (\times t)$

$\Rightarrow 18 = 12t^2 - 15t \quad (\div 3) \Rightarrow 4t^2 - 5t - 6 = 0$ #

b) $(4t+3)(t-2) = 0 \Rightarrow t = -\frac{3}{4}, \; t = 2.$

$t = 2, \; P = \left(6, \frac{3}{2}\right) \qquad t = -\frac{3}{4}, \; Q = \left(-\frac{9}{4}, -4\right)$

6. $$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

The transformation represented by **B** followed by the transformation represented by **A** is equivalent to the transformation represented by **P**.

(a) Find the matrix **P**.

(2)

Triangle T is transformed to the triangle T' by the transformation represented by **P**.

Given that the area of triangle T' is 24 square units,

(b) find the area of triangle T.

(3)

Triangle T' is transformed to the original triangle T by the matrix represented by **Q**.

(c) Find the matrix **Q**.

(2)

a) $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -2 & -3 \end{pmatrix}$

b) $\det P = 1 \times -3 \; - 4 \times -2 = -3 + 8 = 5$

area of $T \times 5$ = area of T' $\therefore$ area of $T = 4.8$

c) $Q = P^{-1} = \frac{1}{5}\begin{pmatrix} -3 & -4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

7. The parabola C has equation $y^2 = 4ax$, where a is a positive constant.

The point $P(at^2, 2at)$ is a general point on C.

(a) Show that the equation of the tangent to C at $P(at^2, 2at)$ is

$$ty = x + at^2$$

(4)

The tangent to C at P meets the y-axis at a point Q.

(b) Find the coordinates of Q.

(1)

Given that the point S is the focus of C,

(c) show that PQ is perpendicular to SQ.

(3)

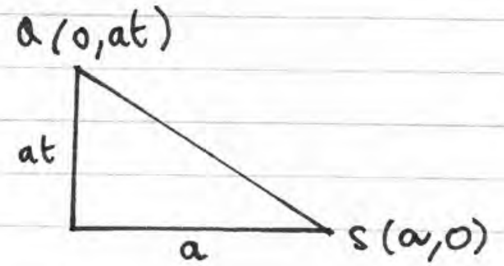
$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \qquad m_t\Big|_{y=2at} = \frac{2a}{2at} = \frac{1}{t}$

$(at^2, 2at)$: $\quad y - 2at = \frac{1}{t}(x - at^2) \Rightarrow ty - 2at^2 = x - at^2$

$\therefore ty = x + at^2$ #

b) $x = 0,\ \cancel{t}y = at^{\cancel{2}} \Rightarrow y = at \qquad Q(0, at)$

c) $S(a, 0)$



$\therefore\ m = \frac{-at}{a} = -t$

$m_{QS} \times m_{PQ} = -t \times \frac{1}{t} = -1$

$\therefore$ QS and PQ are perp.

8. (a) Prove by induction, that for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r(2r-1) = \frac{1}{6}n(n+1)(4n-1)$$

(6)

(b) Hence, show that

$$\sum_{r=n+1}^{3n} r(2r-1) = \frac{1}{3}n(an^2 + bn + c)$$

where a, b and c are integers to be found.

(4)

a) $\sum r(2r-1) = 1(2-1) = 1$ when $n=1$

$\frac{1}{6}n(n+1)(4n-1) = \frac{1}{6}(1)(2)(3) = 1$ when $n=1$ $\quad \therefore$ true for $n=1$

assume true when $n=k$.

$$\sum_{1}^{k+1} r(2r-1) = \frac{1}{6}(k+1)(k+1+1)(4(k+1)-1)$$

$$= \frac{1}{6}(k+1)(k+2)(4k+3)$$

$$\sum_{1}^{k+1} r(2r-1) = (k+1)(2(k+1)-1) + \sum_{1}^{k} r(2r-1)$$

$$= (k+1)(2k+1) + \frac{1}{6}k(k+1)(4k-1)$$

$$= \frac{1}{6}(k+1)\left[6(2k+1) + k(4k-1)\right]$$

$$= \frac{1}{6}(k+1)\left[4k^2 + 11k + 6\right] = \frac{1}{6}(k+1)(k+2)(4k+3) \quad \#$$

$\therefore$ true for $n=1$, true for $n=k+1$ if true for $n=k$

$\therefore$ by Induction true for all $n \in \mathbb{Z}^+$

b) $$\sum_{r=n+1}^{3n} r(2r-1) = \sum_{r=1}^{3n} r(2r-1) - \sum_{r=1}^{n} r(2r-1)$$

$$= \frac{1}{6}(3n)(3n+1)(4(3n)-1) - \frac{1}{6}n(n+1)(4n-1)$$

$$= \frac{1}{6}(3n)(3n+1)(12n-1) - \frac{1}{6}n(n+1)(4n-1)$$

$$= \frac{1}{6}n\left[(9n+3)(12n-1) - (n+1)(4n-1)\right]$$

$$= \frac{1}{6}n\left[108n^2 + 27n - 3 - 4n^2 - 3n + 1\right]$$

$$= \frac{1}{6}n\left[104n^2 + 24n - 2\right]$$

$$= \frac{2}{6}n\left[52n^2 + 12n - 1\right]$$

$$= \frac{1}{3}n(52n^2 + 12n - 1)$$

Leave blank

9. The complex number w is given by

$$w = 10 - 5\mathrm{i}$$

(a) Find $|w|$.

(1)

(b) Find arg w, giving your answer in radians to 2 decimal places.

(2)

The complex numbers z and w satisfy the equation

$$(2 + \mathrm{i})(z + 3\mathrm{i}) = w$$

(c) Use algebra to find z, giving your answer in the form $a + b\mathrm{i}$, where a and b are real numbers.

(4)

Given that

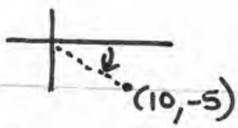
$$\arg(\lambda + 9\mathrm{i} + w) = \frac{\pi}{4}$$

where λ is a real constant,

(d) find the value of λ.

(2)

a) $|w| = \sqrt{10^2 + 5^2} = \sqrt{125}$

b) 

$\arg w = -\tan^{-1}\left(\frac{5}{10}\right)$

$\therefore \arg w = -0.46$

c) $(2+i)(a+(3+b)i) = 2a + (a+6+2b)i - (3+b)$

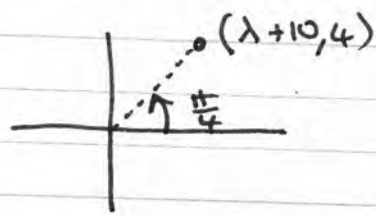
$\Rightarrow (2a-3-b) + (a+6+2b)i \equiv 10 - 5i$

$\Rightarrow 2a - b = 13$
$a + 2b = -11$

$\Rightarrow 4a - 2b = 26$
$a + 2b = -11$
$5a = 15$

$\therefore a = 3$
$b = -7$

d) $\arg(\lambda + 9i + 10 - 5i)$
$= \arg(\lambda + 10 + 4i) = \frac{\pi}{4}$



$\therefore \lambda + 10 = 4 \Rightarrow \lambda = -6$

10. (i) Use the standard results for $\sum_{r=1}^{n} r^3$ and $\sum_{r=1}^{n} r$ to evaluate

$$\sum_{r=1}^{24}(r^3 - 4r)$$

(2)

(ii) Use the standard results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that

$$\sum_{r=0}^{n}(r^2 - 2r + 2n + 1) = \frac{1}{6}(n+1)(n+a)(bn+c)$$

for all integers $n \geqslant 0$, where a, b and c are constant integers to be found.

(6)

a) $\sum_{1}^{24} r^3 - 4\sum_{1}^{24} r = \frac{1}{4}(24)^2(24+1)^2 - 4 \times \frac{1}{2}(24)(24+1)$

$= 90000 - 1200 = \underline{88800}$

b) $\sum_{0}^{n} r^2-2r+2n+1 = (2n+1)+\sum_{1}^{n} r^2-2r+2n+1$

$= (2n+1)+\sum r^2 - 2\sum r + (2n+1)\sum 1$

$= (2n+1) + \frac{1}{6}n(n+1)(2n+1) - 2\times\frac{1}{2}n(n+1) + (2n+1)\times n$

$= \frac{1}{6}[n(n+1)(2n+1) + 6(2n+1) - 6n(n+1) + 6n(2n+1)]$

$= \frac{1}{6}[(n+1)[n(2n+1)-6n] + 6(2n+1)(n+1)]$

$= \frac{1}{6}[(n+1)(n(2n+1)-6n+6(2n+1))]$

$= \frac{1}{6}[(n+1)(2n^2+n-6n+12n+6)]$

$= \frac{1}{6}[(n+1)(2n^2+7n+6)]$

$= \frac{1}{6}(n+1)(n+2)(2n+3)$