FP1 2013 Withdrawn (MA)

Q1a) $\det M = a(2-a) - 1 = \boxed{2a - a^2 - 1}$

b) $\text{Area}_{\text{image}} = \text{Area}_{\text{obj}} \times |\det M|$

$0 = \text{Area}_{\text{obj}} \times |\det M|$

$\therefore |\det M| = 0$

$\Rightarrow a^2 - 2a + 1 = 0$

$\Rightarrow (a-1)^2 = 0$

$\boxed{a = 1}$

Q2) $-1-2i$ will be another root

$(z-(-1-2i))(z-(-1+2i)) = (z^2+2z+5)$

$\therefore f(z) = (z^2+2z+5)(z-a) = 0$

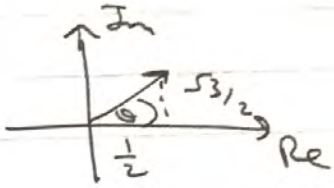
$\Rightarrow (z^2+2z+5)(z+3) = 0$

$\therefore \boxed{z = -3}$ and $z = -1-2i$
$z = -1+2i$

Q3a) $z_1 = \frac{1}{2}(1+\sqrt{3}i)$

$|z_1| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} = 1$ //

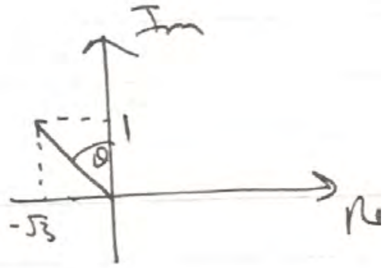
$\arg z_1 = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} = \theta$ //



$\therefore z_1 = \boxed{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

$z_2 = -\sqrt{3} + i$

$|z_2| = \sqrt{(\sqrt{3})^2+(1)^2} = 2$ //



$\tan\theta = \frac{1}{\frac{1}{\sqrt{3}}} \quad \therefore \theta = \tan^{-1}(\sqrt{3})$
$= \frac{\pi}{3}$

$\therefore \arg z_2 = \frac{\pi}{3} + \frac{\pi}{2} = \frac{5\pi}{6}$ //

$\therefore z_2 = \boxed{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

b) $z_1 z_2 = \frac{1}{2}(1+\sqrt{3}i)(-\sqrt{3}+i)$

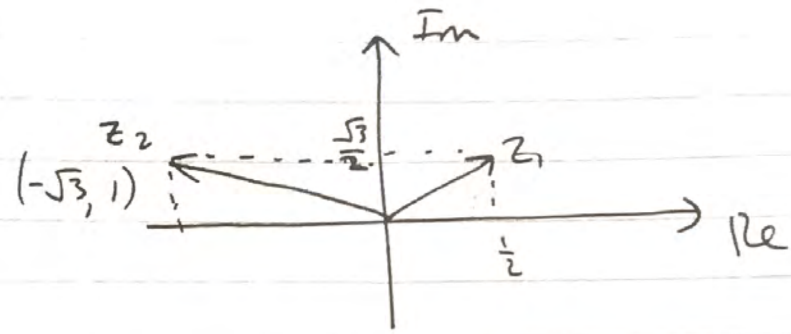
$= \frac{1}{2}\left(-\sqrt{3} + i - 3i - \sqrt{3}\right)$

$= \frac{1}{2}\left(-2\sqrt{3} - 2i\right) = -\sqrt{3} - i$

$\therefore |z_1 z_2| = \sqrt{(-\sqrt{3})^2+(-1)^2} = \boxed{2}$

alt: from FP2, $|z_1 z_2| = |z_1||z_2| = 2 \times 1 = \boxed{2}$

3c)



Q4a) $xy = 3$.

$y + x\frac{dy}{dx} = 0 \quad \therefore \quad \frac{dy}{dx} = -\frac{y}{x} = -\frac{3}{1} = -3$

so at normal, $m = \frac{1}{3}$ $\left(-3 \times \frac{1}{3} = -1\right)$

$\Rightarrow y - 3 = \frac{1}{3}(x - 1)$

$\Rightarrow y = \frac{1}{3}x + \frac{8}{3}$

b) $x\left(\frac{1}{3}x + \frac{8}{3}\right) = 3$

$\times 3$: $x(x + 8) = 9$

$\Rightarrow x^2 + 8x - 9 = 0$

$\Rightarrow (x + 9)(x - 1) = 0$

$x = -9 \rightarrow y = \frac{1}{3}(-9) + \frac{8}{3} = -\frac{1}{3}$

$\therefore \boxed{R\left(-9, -\frac{1}{3}\right)}$

Q5) $f(n) = 3^{2n} + 7$

$\underline{n=1}$: $f(1) = 3^2 + 7 = 16 = (8) \times 2$

$\therefore$ true for $n = 1$

$\underline{\text{assume true for } n=k}$,

(ie) $[f(k) = 3^{2k} + 7]$ is divisible by 8.

$\underline{\text{consider } n=k+1}$,

$f(k+1) = 3^{2(k+1)} + 7 = 3^{2k} \times 3^2 + 7$

$= 9(3^{2k}) + 7$

$= 9[3^{2k} + 7] - 8(7)$

$= 9f(k) - 8(7)$

$\therefore$ true for $n = k+1$

so, true for $n = 1$.

true for $n = k+1$ when assumed true for $n = k$.

$\therefore$ By Mathematical Induction true for all $n \in \mathbb{Z}^+$

Q6a) $y^2 = 4x$

$2y\frac{dy}{dx} = 4 \longrightarrow \frac{dy}{dx} = \frac{2}{y} = \frac{2}{2p} = \frac{1}{p}$ //

$\Rightarrow y - 2p = \frac{1}{p}(x - p^2)$ //

$\Rightarrow y = \frac{1}{p}x - p + 2p$

$\Rightarrow py = x + p^2$ //

bi) $R(-1, 2)$: $2p = -1 + p^2$

$p^2 - 2p - 1 = 0$

By Quadratic formula: $(p = 1 \pm \sqrt{2})$ //

$\hookrightarrow \boxed{p = 1 + \sqrt{2}}$

tangent at P gradient × tangent at Q gradient

$\frac{1}{p} \times \frac{1}{q} = -1$

$\therefore pq = -1$ //

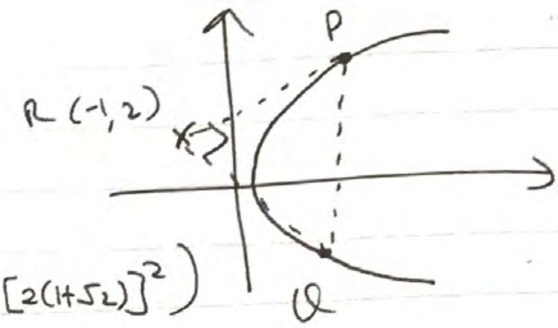
but $pq = -1$ (they are perpendicular)

so $q = -\frac{1}{p} = -\frac{1}{1+\sqrt{2}} \frac{\times(1-\sqrt{2})}{\times(1-\sqrt{2})}$

$= \boxed{1 - \sqrt{2}}$

ii) $\text{Area} = \frac{1}{2} bh$

$= \frac{1}{2}(PR)(QR)$



$PR^2 = (-1-(1+\sqrt{2})^2) + (2-[2(1+\sqrt{2})]^2)$

$= 32 + 16\sqrt{2}$

$QR^2 = (-1-(1-\sqrt{2})^2) + (2-[2(1-\sqrt{2})]^2)$

$= 32 - 16\sqrt{2}$

$\therefore \text{Area} = \frac{1}{2}(\sqrt{32+16\sqrt{2}})(\sqrt{32-16\sqrt{2}})$

$= \frac{1}{2}\sqrt{32^2 - (16\sqrt{2})^2} = \frac{1}{2}\sqrt{1024-512}$

$= \frac{1}{2} \times 16\sqrt{2}$

$= \boxed{8\sqrt{2}}$

Q7a) $\sum_1^n r^2(r-1) = \sum_1^n r^3 - r^2 = \sum_1^n r^3 - \sum_1^n r^2$

$= \frac{n^2}{4}(n+1)^2 - \frac{n}{6}(n+1)(2n+1)$

$= \frac{n}{12}(n+1)\left[3n(n+1) - 2(2n+1)\right]$

$= \frac{n}{12}(n+1)\left[3n^2 + 3n - 4n - 2\right]$

$= \frac{n}{12}(n+1)(3n^2 - n - 2)$

$= \frac{n}{12}(n+1)(3n+2)(n-1) = \underline{\frac{n(n+1)(3n+2)(n-1)}{12}}$

b) sum required $= \sum_{10}^{50} r^2(r-1)$

$= \sum_{1}^{50} r^2(r-1) - \sum_{1}^{9} r^2(r-1)$

$= \frac{50(51)(152)(49)}{12} - \frac{9(10)(29)(8)}{12}$

$= 1582700 - 1740$

$= \boxed{1580960}$

Q8a) $f(1) = 1 - 2 - 3 = -4$

$f(2) = 8 - 4 - 3 = 1$

change in sign between $x=1$ and $x=2$ $\therefore$ a root lies in this interval.

b)

a	$f(a)$	b	$f(b)$	$\frac{a+b}{2}$	$f\left(\frac{a+b}{2}\right)$
1	-4	2	1	1.5	$-21/8$
1.5	$-21/8$	2	1	1.75	$-73/64$

$\therefore$ interval : $[1.75, 2]$ //

c) $f'(x) = 3x^2 - 2$

$f(1.8) = -96/125$

$f'(1.8) = 193/25$

$x_1 = 1.8 - \frac{193/25}{-96/125}$

$\simeq \boxed{1.90}$ 3sf.

Q9a) $A^2 = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 9+1 & 3-2 \\ 3-2 & 4+1 \end{pmatrix}$

$$= \boxed{\begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix}}$$

b) $\det A = 3(-2) - 1 = -7 \neq 0$

hence A is non-singular.

c) $A^{-1} = -\frac{1}{7}\begin{pmatrix} -2 & -1 \\ -1 & 3 \end{pmatrix}$

d) $AP = Q$.

$$\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k-1 \\ 2-k \end{pmatrix}$$

$3x + y = k - 1$

$y = k - 1 - 3x \sim (1)$

$x - 2y = 2 - k$

$x = 2y + 2 - k \sim (2)$

so (1): $y = k - 1 - 3(2y + 2 - k)$

$y = k - 1 - 6y - 6 + 3k$

$y = -6y + 4k - 7$

$7y = 4k - 7 \rightarrow y = \frac{4}{7}k - 1$ //

so $x = 2\left(\frac{4}{7}u - 1\right) + 2 - u$

$x = \frac{8}{7}u - u = \frac{u}{7}$ //

hence $y = 4\left(\frac{1}{7}u\right) - 1$

$y = 4x - 1$

$\square$