

$$f(x) = 2x^3 - 6x^2 - 7x - 4$$

(a) Show that  $f(4) = 0$

(1)

(b) Use algebra to solve  $f(x) = 0$  completely.

(4)

$$\begin{aligned} \text{a) } f(4) &= 2(4)^3 - 6(4)^2 - 7(4) - 4 \\ &= 128 - 96 - 28 - 4 \\ &= 0 \quad \# \end{aligned}$$

$\therefore (x-4)$  is a factor of  $f(x)$ .

$$\begin{array}{r} \text{b) } \quad \quad \quad 2x^2 + 2x + 1 \\ x \quad \begin{array}{|c|c|c|} \hline 2x^3 & +2x^2 & +x \\ \hline -8x^2 & -8x & -4 \\ \hline \end{array} \quad r=0 \checkmark \\ -4 \end{array}$$

$$2x^2 + 2x + 1 = 0$$

$$\textcircled{\div 2} \quad x^2 + x = -\frac{1}{2}$$

$$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} = -\frac{1}{2}$$

$$\left(x + \frac{1}{2}\right)^2 = -\frac{1}{4}$$

$$\Rightarrow x + \frac{1}{2} = \pm \sqrt{-\frac{1}{4}}$$

$$\Rightarrow x + \frac{1}{2} = \pm \frac{1}{2}i \quad \Rightarrow x = -\frac{1}{2} \pm \frac{1}{2}i$$

Solutions are  $4, -\frac{1}{2} + \frac{1}{2}i, -\frac{1}{2} - \frac{1}{2}i$

2. (a) Given that

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 4 & 5 & 5 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}$$

find  $AB$ .

(2)

(b) Given that

$$C = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix}, \quad \text{where } k \text{ is a constant}$$

and

$$E = C + D$$

find the value of  $k$  for which  $E$  has no inverse.

(4)

$$a) AB = \begin{pmatrix} 3 & 1 & 3 \\ 4 & 5 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 9 & 9 \end{pmatrix}$$

$$b) C + D = \begin{pmatrix} 3 & 2 \\ 8 & 6 \end{pmatrix} + \begin{pmatrix} 5 & 2k \\ 4 & k \end{pmatrix} = \begin{pmatrix} 8 & 2+2k \\ 12 & 6+k \end{pmatrix} = E$$

$$\text{no inverse} \Rightarrow \det(E) = 0$$

$$\Rightarrow 8(6+k) - 12(2+2k) = 0$$

$$\Rightarrow 48 + 8k - 24 - 24k = 0$$

$$\Rightarrow 24 = 16k \Rightarrow k = \frac{3}{2}$$

3.

$$f(x) = x^2 + \frac{3}{4\sqrt{x}} - 3x - 7, \quad x > 0$$

A root  $\alpha$  of the equation  $f(x) = 0$  lies in the interval  $[3, 5]$ .

Taking 4 as a first approximation to  $\alpha$ , apply the Newton-Raphson process once to  $f(x)$  to obtain a second approximation to  $\alpha$ . Give your answer to 2 decimal places.

(6)

$$x_0 = 4 \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f(4) = 16 + \frac{3}{8} - 12 - 7 = -\frac{21}{8}$$

$$f(x) = x^2 + \frac{3}{4}x^{-\frac{1}{2}} - 3x - 7$$

$$f'(x) = 2x - \frac{3}{8}x^{-\frac{3}{2}} - 3$$

$$f'(4) = 8 - \frac{3}{64} - 3 = \frac{317}{64}$$

$$\therefore x_1 = 4 - \frac{-\frac{21}{8}}{\frac{317}{64}} = \frac{1436}{317} = \underline{\underline{4.53}}$$

4. (a) Use the standard results for  $\sum_{r=1}^n r^3$  and  $\sum_{r=1}^n r$  to show that

$$\sum_{r=1}^n (r^3 + 6r - 3) = \frac{1}{4}n^2(n^2 + 2n + 13)$$

for all positive integers  $n$ .

(5)

- (b) Hence find the exact value of

$$\sum_{r=16}^{30} (r^3 + 6r - 3)$$

(2)

$$\sum (r^3 + 6r - 3) = \sum r^3 + 6 \sum r - 3 \sum 1$$

$$= \frac{1}{4}n^2(n+1)^2 + \frac{6}{2}n(n+1) - 3n$$

$$= \frac{1}{4}n [n(n+1)^2 + 12(n+1) - 12]$$

$$= \frac{1}{4}n [n(n+1)^2 + 12n]$$

$$= \frac{1}{4}n^2 [(n+1)^2 + 12]$$

$$= \frac{1}{4}n^2 [n^2 + 2n + 13] \quad \#$$

$$b) \sum_{16}^{30} (r^3 + 6r - 3) = \left[ \frac{1}{4}n^2 [n^2 + 2n + 13] \right]_{16}^{30}$$

$$= (218925) - (15075)$$

$$= 203850$$

5.

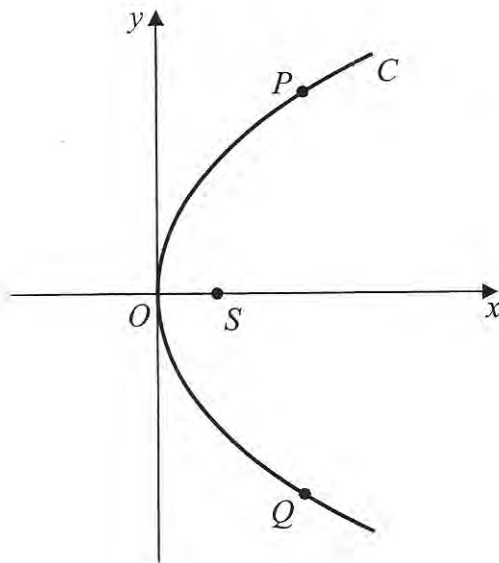


Figure 1

Figure 1 shows a sketch of the parabola  $C$  with equation  $y^2 = 8x$ .  
 The point  $P$  lies on  $C$ , where  $y > 0$ , and the point  $Q$  lies on  $C$ , where  $y < 0$ .  
 The line segment  $PQ$  is parallel to the  $y$ -axis.

Given that the distance  $PQ$  is 12,

- (a) write down the  $y$ -coordinate of  $P$ , (1)
- (b) find the  $x$ -coordinate of  $P$ . (2)

Figure 1 shows the point  $S$  which is the focus of  $C$ .  
 The line  $l$  passes through the point  $P$  and the point  $S$ .

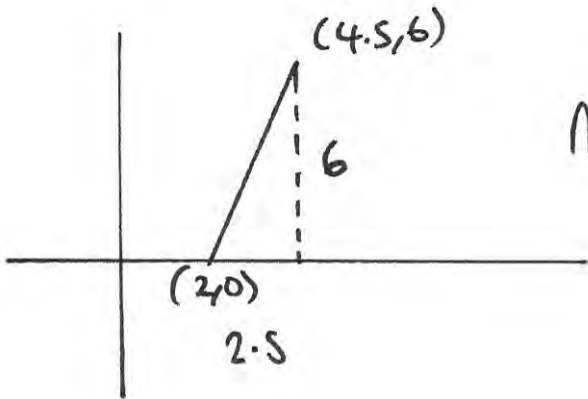
- (c) Find an equation for  $l$  in the form  $ax + by + c = 0$ , where  $a$ ,  $b$  and  $c$  are integers. (4)

$$y^2 = 4ax \quad \therefore a = 2 \quad \therefore \text{focus } (2, 0) \\ \text{directrix } x + 2 = 0$$

$$\therefore x = 4t^2 \quad y = 4t$$

$$\text{a) } PQ = 12 \Rightarrow y \text{ coordinate of } P = 6$$

$$\text{b) } y^2 = 8x \Rightarrow 36 = 8x \Rightarrow x = 4.5$$



$$m = \frac{6}{2.5}$$

$$y - y_1 = m(x - x_1)^{\text{PMT}}$$

$$y = \frac{6}{2.5}(x - 2)$$

$$2.5y = 6x - 12$$

$$\underline{12x - 5y - 24 = 0}$$

6.

$$f(x) = \tan\left(\frac{x}{2}\right) + 3x - 6, \quad -\pi < x < \pi$$

(a) Show that the equation  $f(x) = 0$  has a root  $\alpha$  in the interval  $[1, 2]$ .

(2)

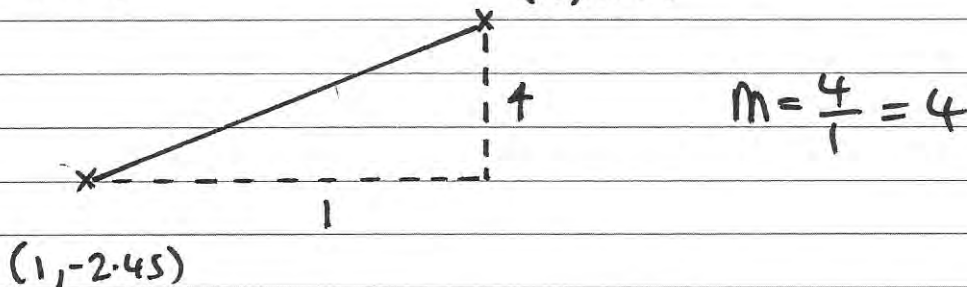
(b) Use linear interpolation once on the interval  $[1, 2]$  to find an approximation to  $\alpha$ .  
Give your answer to 2 decimal places.

(3)

$$f(1) = -2.454 \quad \therefore \text{by change of sign rule}$$

$$f(2) = 1.557 \quad \alpha \in (1, 2)$$

(2, 1.56)



$$y - y_1 = m(x - x_1)$$

$$y - 1.557 = 4(x - 2)$$

$$\Rightarrow \text{when } y = 0 \quad -1.557 = 4(\alpha - 2)$$

$$-0.389 = \alpha - 2$$

$$\alpha = 1.61$$

7.

$$z = 2 - i\sqrt{3}$$

(a) Calculate  $\arg z$ , giving your answer in radians to 2 decimal places.

(2)

Use algebra to express

(b)  $z + z^2$  in the form  $a + bi\sqrt{3}$ , where  $a$  and  $b$  are integers,

(3)

(c)  $\frac{z+7}{z-1}$  in the form  $c + di\sqrt{3}$ , where  $c$  and  $d$  are integers.

(4)

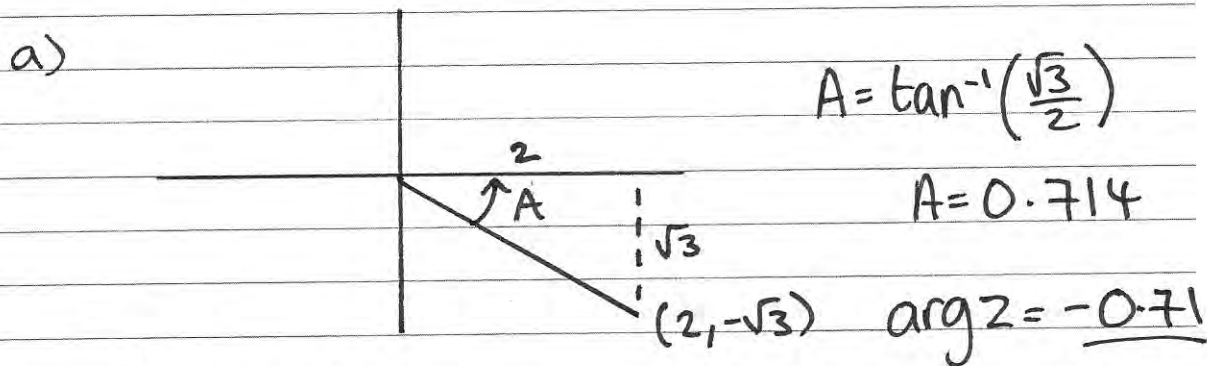
Given that

$$w = \lambda - 3i$$

where  $\lambda$  is a real constant, and  $\arg(4 - 5i + 3w) = -\frac{\pi}{2}$ ,

(d) find the value of  $\lambda$ .

(2)



b)

$$(2 - i\sqrt{3}) + (2 - i\sqrt{3})^2$$

$$= (2 - i\sqrt{3}) + (1 - i4\sqrt{3})$$

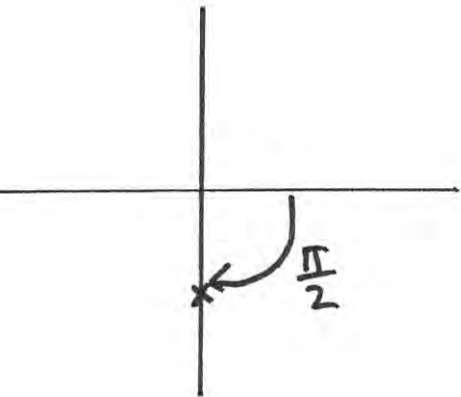
$$= 3 - i5\sqrt{3}$$

c)

$$\frac{z+7}{z-1} = \frac{9 - i\sqrt{3}}{1 - i\sqrt{3}} \times \frac{(1 + i\sqrt{3})}{(1 + i\sqrt{3})} = \frac{12 + i8\sqrt{3}}{4}$$

$$= \underline{3 + i2\sqrt{3}}$$





$$4 - 5i + 3\omega$$

$$4 - 5i + 3(\lambda - 3i)$$

$$(4 + 3\lambda) - 14i$$

$$\therefore 4 + 3\lambda = 0$$

$$\lambda = -\frac{4}{3}$$

8. The rectangular hyperbola  $H$  has equation  $xy = c^2$ , where  $c$  is a positive constant.

The point  $P\left(ct, \frac{c}{t}\right)$ ,  $t \neq 0$ , is a general point on  $H$ .

- (a) Show that an equation for the tangent to  $H$  at  $P$  is

$$x + t^2y = 2ct \quad (4)$$

The tangent to  $H$  at the point  $P$  meets the  $x$ -axis at the point  $A$  and the  $y$ -axis at the point  $B$ .

Given that the area of the triangle  $OAB$ , where  $O$  is the origin, is 36,

- (b) find the exact value of  $c$ , expressing your answer in the form  $k\sqrt{2}$ , where  $k$  is an integer.

(4)

$$8. \quad xy = c^2 \Rightarrow y = c^2 x^{-1}$$

$$\frac{dy}{dx} = -c^2 x^{-2} = -\left(\frac{c}{x}\right)^2$$

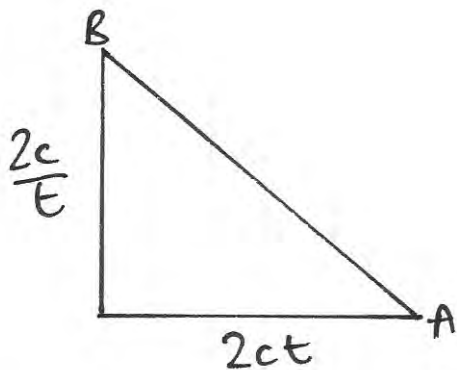
$$P\left(ct, \frac{c}{t}\right) \Rightarrow M_t = -\left(\frac{c}{ct}\right)^2 = -\left(\frac{1}{t}\right)^2 = -\frac{1}{t^2}$$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct) \Rightarrow t^2 y - ct = -x + ct$$

$$\Rightarrow x + t^2 y = 2ct \quad \#$$

$$b) \quad x=0 \Rightarrow y = \frac{2ct}{t^2} \Rightarrow \left(0, \frac{2c}{t}\right) B$$

$$y=0 \Rightarrow x = 2ct \Rightarrow (2ct, 0) A$$



$$\text{Area} = \frac{2ct \left(\frac{2c}{t}\right)}{2} = 36$$

$$\therefore 2c^2 = 36 \quad \therefore c^2 = 18$$

$$c = \underline{3\sqrt{2}}$$

$$\mathbf{M} = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix}$$

9. (a) Find  $\det \mathbf{M}$ .

(1)

The transformation represented by  $\mathbf{M}$  maps the point  $S(2a - 7, a - 1)$ , where  $a$  is a constant, onto the point  $S'(25, -14)$ .

(b) Find the value of  $a$ .

(3)

The point  $R$  has coordinates  $(6, 0)$ .

Given that  $O$  is the origin,

(c) find the area of triangle  $ORS$ .

(2)

Triangle  $ORS$  is mapped onto triangle  $OR'S'$  by the transformation represented by  $\mathbf{M}$ .

(d) Find the area of triangle  $OR'S'$ .

(2)

Given that

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(e) describe fully the single geometrical transformation represented by  $\mathbf{A}$ .

(2)

The transformation represented by  $\mathbf{A}$  followed by the transformation represented by  $\mathbf{B}$  is equivalent to the transformation represented by  $\mathbf{M}$ .

(f) Find  $\mathbf{B}$ .

(4)

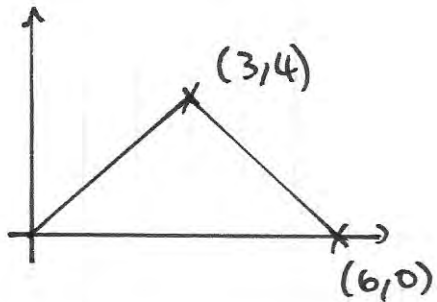
$$a) \det M = -15 - 8 = -23$$

$$b) \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 2a-7 \\ a-1 \end{pmatrix} = \begin{pmatrix} 25 \\ -14 \end{pmatrix}$$

$$\therefore 3(2a-7) + 4(a-1) = 25$$

$$6a - 21 + 4a - 4 = 25 \Rightarrow 10a = 50 \Rightarrow \underline{a=5}$$

c)

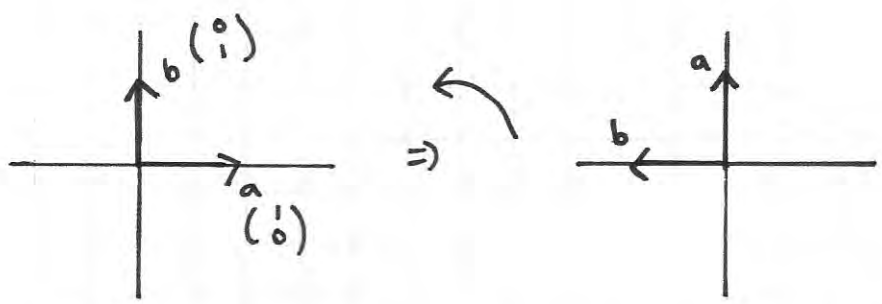


$$a=5 \Rightarrow S(3,4)$$

$$\text{Area} = \frac{6 \times 4}{2} = 12$$

d)  $\det(M) = -23 \quad \therefore \text{AREA OR}'S' = 12 \times 23 = 276$

e)



rotation  $90^\circ$  anticlockwise about  $(0,0)$

f)  $BA = M \Rightarrow BAA^{-1} = MA^{-1}$

$$\Rightarrow B = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} A^{-1}$$

$$\det(A) = 1 \Rightarrow A^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 4 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 5 & 2 \end{pmatrix}$$

10. Prove by induction that, for  $n \in \mathbb{Z}^+$ ,

$$f(n) = 2^{2n-1} + 3^{2n-1} \text{ is divisible by } 5.$$

(6)

$$n=1 \quad f(1) = 2^1 + 3^1 = 5 \quad \text{true for } n=1$$

$$n=2 \quad f(2) = 2^3 + 3^3 = 35 = 7 \times 5 \quad \text{true for } n=2$$

assume true for  $n=k$

$$\therefore f(k) = 2^{2k-1} + 3^{2k-1} \text{ is divisible by } 5$$

$n=k+1$

$$f(k+1) = 2^{2(k+1)-1} + 3^{2(k+1)-1} = 2^{2k+1} + 3^{2k+1}$$

$$= 2^{(2k-1)+2} + 3^{(2k-1)+2}$$

$$= 2^2(2^{2k-1}) + 3^2(3^{2k-1})$$

$$= 4[2^{2k-1} + 3^{2k-1}] + 5(3^{2k-1})$$

$$= 4f(k) + 5(3^{2k-1})$$

Therefore, if  $f(k)$  is divisible by 5  $f(k+1)$  must also be divisible by 5

$\therefore$  true for  $n=1, n=2$

true for  $n=k+1$  if true for  $n=k$

$\therefore$  by induction true for all  $n \geq 1 \quad n \in \mathbb{Z}^+$