

1.

$$f(x) = 3^x + 3x - 7$$

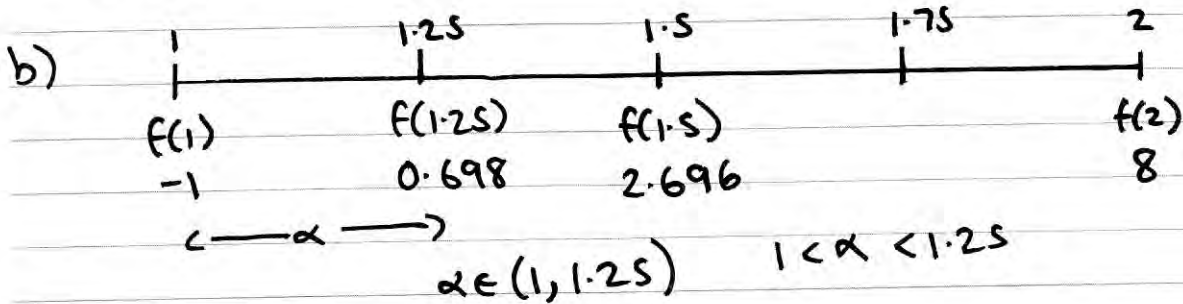
(a) Show that the equation $f(x) = 0$ has a root α between $x = 1$ and $x = 2$.

(2)

(b) Starting with the interval $[1, 2]$, use interval bisection twice to find an interval of width 0.25 which contains α .

(3)

a) $f(1) = -1$ change of sign \Rightarrow $1 < \alpha < 2$
 $f(2) = 8$ $\alpha \in (1, 2)$



2.

$$z_1 = -2 + i$$

(a) Find the modulus of z_1 .

(1)

(b) Find, in radians, the argument of z_1 , giving your answer to 2 decimal places.

(2)

The solutions to the quadratic equation

$$z^2 - 10z + 28 = 0$$

are z_2 and z_3 .(c) Find z_2 and z_3 , giving your answers in the form $p \pm i\sqrt{q}$, where p and q are integers.

(3)

(d) Show, on an Argand diagram, the points representing your complex numbers z_1 , z_2 and z_3 .

(2)

$$a) |z_1| = |-2 + i| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

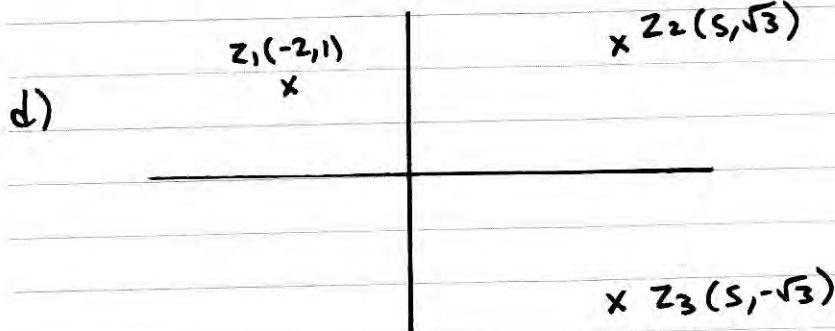
$$b) \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\sqrt{5} \times 1}{\sqrt{2} \times -2} \Rightarrow \theta = \tan^{-1}\left(-\frac{1}{2}\right) = -0.4636 \dots$$

$$+ \pi \quad \arg(z_1) = \underline{2.68^\circ}$$

$$c) (z - 5)^2 - 28 + 28 = 0$$

$$\Rightarrow (z - 5)^2 = -3 \Rightarrow z - 5 = \pm\sqrt{-3} \Rightarrow z = 5 \pm i\sqrt{3}$$

$$\therefore z_2 = 5 + i\sqrt{3} \quad z_3 = 5 - i\sqrt{3}$$



3. (a) Given that

$$\mathbf{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}$$

(i) find \mathbf{A}^2 ,

(ii) describe fully the geometrical transformation represented by \mathbf{A}^2 .

(4)

(b) Given that

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

describe fully the geometrical transformation represented by \mathbf{B} .

(2)

(c) Given that

$$\mathbf{C} = \begin{pmatrix} k+1 & 12 \\ k & 9 \end{pmatrix}$$

where k is a constant, find the value of k for which the matrix \mathbf{C} is singular.

(3)

i) $\mathbf{A}^2 = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ii) enlargement
S.f 3 centre origin.

b) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ reflection through line $y = -x$

c) $\det \mathbf{C} = 9(k+1) - 12k = 9 - 3k$

$\det \mathbf{C} = 0$ if singular $\therefore \underline{k = 3}$

$$f(x) = x^2 + \frac{5}{2x} - 3x - 1, \quad x \neq 0$$

4.

(a) Use differentiation to find $f'(x)$.

(2)

The root α of the equation $f(x) = 0$ lies in the interval $[0.7, 0.9]$.

(b) Taking 0.8 as a first approximation to α , apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α . Give your answer to 3 decimal places.

(4)

$$a) f(x) = x^2 + \frac{5}{2}x^{-1} - 3x - 1$$

$$f'(x) = 2x - \frac{5}{2}x^{-2} - 3$$

$$x_1 = 0.8 \quad x_2 = 0.8 - \frac{f(0.8)}{f'(0.8)} = 0.8 - \frac{\frac{73}{200}}{-\frac{849}{160}}$$

$$x_2 \approx \underline{0.869} \quad (3dp)$$

5.

$$\mathbf{A} = \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix}, \text{ where } a \text{ and } b \text{ are constants.}$$

Given that the matrix \mathbf{A} maps the point with coordinates $(4, 6)$ onto the point with coordinates $(2, -8)$,

(a) find the value of a and the value of b .

(4)

A quadrilateral R has area 30 square units.

It is transformed into another quadrilateral S by the matrix \mathbf{A} .

Using your values of a and b ,

(b) find the area of quadrilateral S .

(4)

$$\text{a) } \begin{pmatrix} -4 & a \\ b & -2 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix} \Rightarrow \begin{aligned} -16 + 6a &= 2 \Rightarrow a = 3 \\ 4b - 12 &= -8 \Rightarrow b = 1 \end{aligned}$$

$$\text{b) } \begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix} \quad \det A = 8 - 3 = 5$$

$$\text{Area } S = 5 \times 30 = \underline{150} \text{ square units}$$

6. Given that $z = x + iy$, find the value of x and the value of y such that

$$z + 3iz^* = -1 + 13i$$

where z^* is the complex conjugate of z .

(7)

$$(x + iy) + 3i(x - iy) = x + iy + i3x - 3i^2y$$

$$= (x + 3y) + (3x + y)i = -1 + 13i$$

$$x + 3y = -1 \quad (\times 3)$$

$$3x + y = 13$$

$$3x + 9y = -3$$

$$3x + y = 13$$

$$8y = -16 \Rightarrow \underline{y = -2}$$

$$x - 6 = -1 \Rightarrow \underline{x = 5}$$

7. (a) Use the results for $\sum_{r=1}^n r$ and $\sum_{r=1}^n r^2$ to show that

$$\sum_{r=1}^n (2r-1)^2 = \frac{1}{3}n(2n+1)(2n-1)$$

for all positive integers n .

(6)

(b) Hence show that

$$\sum_{r=n+1}^{3n} (2r-1)^2 = \frac{2}{3}n(an^2 + b)$$

where a and b are integers to be found.

(4)

$$\sum (2r-1)^2 = 4 \sum r^2 - 4 \sum r + \sum 1$$

$$= 4 \left(\frac{1}{6}n(n+1)(2n+1) \right) - 4 \left(\frac{1}{2}n(n+1) \right) + n$$

$$= \frac{1}{3}n(2(n+1)(2n+1) - 6(n+1) + 3)$$

$$= \frac{1}{3}n(4n^2 + 6n + 2 - 6n - 6 + 3) = \frac{1}{3}n(4n^2 - 1)$$

$$= \frac{1}{3}n(2n+1)(2n-1) \quad \#$$

$$\text{b) } \sum_{r=n+1}^{3n} (2r-1)^2 = \frac{1}{3}(3n)(6n+1)(6n-1) - \frac{1}{3}n(2n+1)(2n-1)$$

$$= \frac{1}{3}n(3(36n^2 - 1) - (4n^2 - 1))$$

$$= \frac{1}{3}n[104n^2 - 2] = \frac{2}{3}n[52n^2 - 1]$$

8. The parabola C has equation $y^2 = 48x$.

The point $P(12t^2, 24t)$ is a general point on C .

(a) Find the equation of the directrix of C .

(2)

(b) Show that the equation of the tangent to C at $P(12t^2, 24t)$ is

$$x - ty + 12t^2 = 0$$

(4)

The tangent to C at the point $(3, 12)$ meets the directrix of C at the point X .

(c) Find the coordinates of X .

(4)

$$a) y^2 = 48x \equiv 4ax \Rightarrow a = 12 \quad \text{directrix } x + 12 = 0$$

$$b) \frac{d}{dx} y^2 = \frac{d}{dx} (48x) \Rightarrow 2y \frac{dy}{dx} = 48 \Rightarrow \frac{dy}{dx} = \frac{24}{y}$$

$$\text{at } P \quad m_t = \frac{24}{24t} = \frac{1}{t} \Rightarrow y - 24t = \frac{1}{t}(x - 12t^2)$$

$$ty - 24t^2 = x - 12t^2 \Rightarrow x - ty + 12t^2 = 0 \quad \#$$

$$b) (3, 12) \Rightarrow 3 = 12t^2 \quad t = \frac{1}{2} \quad 24t = 12 \quad t = \frac{1}{2}$$

$$t = \frac{1}{2} \Rightarrow \frac{1}{2}y = x + 3 \Rightarrow y = 2x + 6$$

$$\text{at directrix } x = -12 \Rightarrow y = -24 + 6 = -18 \quad X(-12, -18)$$

9. Prove by induction, that for $n \in \mathbb{Z}^+$,

$$(a) \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix},$$

(6)

(b) $f(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)

$$a) n=1 \quad \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^1 = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \quad n=1 \quad \begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$$

\therefore true for $n=1$

$$n=k+1 \quad \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^k = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3(3^k) & 0 \\ 6(3^k) + 3(3^k) - 3 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 9(3^k) - 3 & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3^{k+1} & 0 \\ 3^{k+2} - 3 & 1 \end{pmatrix} \quad \#$$

\therefore true for $n=1$, true for $n=k+1$ if true for $n=k$

\therefore true for all $n \in \mathbb{Z}^+$ by induction.

$$b) f(1) = 7^{2-1} + 5 = 12 = 12 \times 1 \quad \therefore \text{true for } n=1$$

$$f(k+1) = 7^{2(k+1)-1} + 5 = 7^{2k+1} + 5 = 7^{2k-1+2} + 5$$

$$= 7^2(7^{2k-1}) + 5 = 49(7^{2k-1} + 5) + 240$$

$$\Rightarrow f(k+1) = 49f(k) + 20 \times 12$$

\therefore true for $n=1$, true for $n=k+1$ if true for $n=k$

\therefore true for all $n \in \mathbb{Z}^+$ by induction