FP1 JAN 13

1. Show, using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$, that

$$\sum_{r=1}^{n} 3(2r-1)^2 = n(2n+1)(2n-1), \text{ for all positive integers } n.$$

(5)

$\sum 12r^2 - 12r + 3 = 12\sum r^2 - 12\sum r + 3\sum 1$

$= 12 \times \frac{1}{6}n(n+1)(2n+1) - 12 \times \frac{1}{2}n(n+1) + 3n$

$= n[2(n+1)(2n+1) - 6(n+1) + 3]$

$= n[4n^2 + \cancel{6n} + 2 - \cancel{6n} - 6 + 3]$

$= n[4n^2 - 1] = n(2n+1)(2n-1)$ #

2.

$$z = \frac{50}{3+4\mathrm{i}}$$

Find, in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$,

(a) z,

(2)

(b) z^2.

(2)

Find

(c) $|z|$,

(2)

(d) $\arg z^2$, giving your answer in degrees to 1 decimal place.

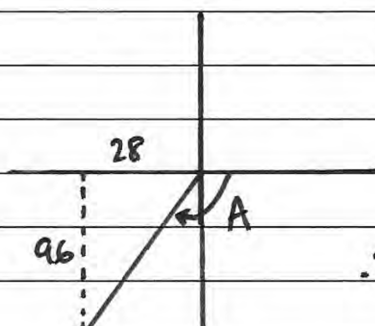
(2)

a) $\frac{50(3-4i)}{(3+4i)(3-4i)} = \frac{150-200i}{9+16} = \frac{150-200i}{25} = \underline{6-8i}$

b) $z^2 = (6-8i)(6-8i) = 36-64-96i = \underline{-28-96i}$

c) $|z| = \sqrt{6^2+8^2} = 10$

d) $\arg z^2$



$A = 180 - \tan^{-1}\left(\frac{96}{28}\right)$

$A = 106.3$

$\therefore \arg z^2 = \underline{-106.3}$

3.

$$f(x) = 2x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 5, \qquad x > 0$$

(a) Find $f'(x)$.

(2)

The equation $f(x) = 0$ has a root α in the interval [4.5, 5.5].

(b) Using $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 significant figures.

(4)

a) $f'(x) = x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$

b) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 5 - \frac{2(5)^{\frac{1}{2}} + (5)^{-\frac{1}{2}} - 5}{(5)^{-\frac{1}{2}} - \frac{1}{2}(5)^{-\frac{3}{2}}}$

$x_1 = 5.200377653..$ $\quad \therefore x_1 = \underline{5.20}$ (3sf)

4. The transformation U, represented by the 2×2 matrix $\mathbf{P}$, is a rotation through $90°$ anticlockwise about the origin.

(a) Write down the matrix $\mathbf{P}$.

(1)

The transformation V, represented by the 2×2 matrix $\mathbf{Q}$, is a reflection in the line $y=-x$.

(b) Write down the matrix $\mathbf{Q}$.

(1)

Given that U followed by V is transformation T, which is represented by the matrix $\mathbf{R}$,

(c) express $\mathbf{R}$ in terms of $\mathbf{P}$ and $\mathbf{Q}$,

(1)

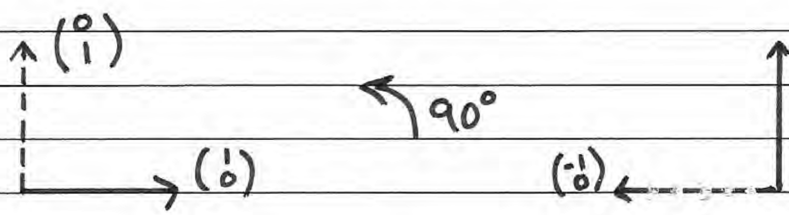
(d) find the matrix $\mathbf{R}$,

(2)

(e) give a full geometrical description of T as a single transformation.

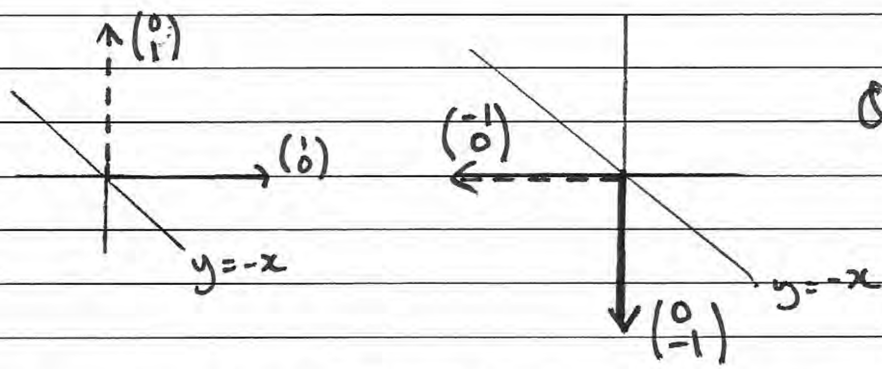
(2)

a)



$\therefore P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b)

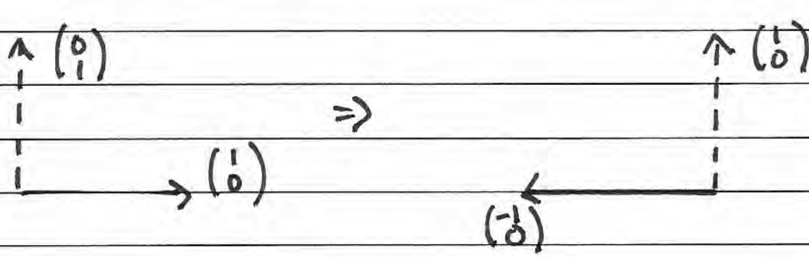


$Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

d) $R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ = (d) $\quad R = VU$

c) $\therefore R = QP$.

e)



reflection through y-axis.

5.

$$f(x) = (4x^2 + 9)(x^2 - 6x + 34)$$

(a) Find the four roots of $f(x) = 0$

Give your answers in the form $x = p + iq$, where p and q are real.

(5)

(b) Show these four roots on a single Argand diagram.

(2)

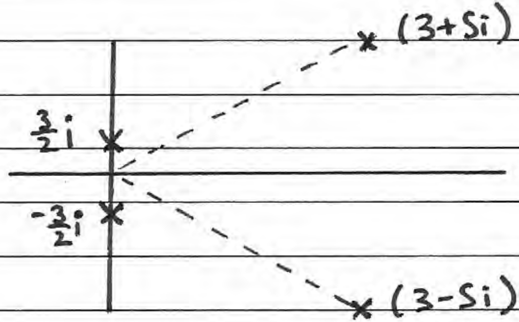
$4x^2 + 9 = 0 \Rightarrow x^2 = -\frac{9}{4} \Rightarrow x = \pm\sqrt{\frac{-9}{4}}$ $\quad x = \frac{3}{2}i, -\frac{3}{2}i$

$x^2 - 6x + 34 = 0 \Rightarrow (x-3)^2 - 9 = -34$

$\Rightarrow (x-3)^2 = -25$

$\Rightarrow x - 3 = \pm 5i \Rightarrow x = 3+5i, 3-5i$

b)



6. $\mathbf{X} = \begin{pmatrix} 1 & a \\ 3 & 2 \end{pmatrix}$, where a is a constant.

(a) Find the value of a for which the matrix **X** is singular.

(2)

$$\mathbf{Y} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$$

(b) Find $\mathbf{Y}^{-1}$.

(2)

The transformation represented by **Y** maps the point A onto the point B.

Given that B has coordinates $(1 - \lambda, 7\lambda - 2)$, where λ is a constant,

(c) find, in terms of λ, the coordinates of point A.

(4)

a) $\det X = 2 - 3a = 0 \quad \therefore \quad a = \frac{2}{3}$

b) $\det Y = 2 - -3 = 5$

$\therefore \quad Y^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{pmatrix}$

c) $YA = B \Rightarrow Y^{-1}YA = Y^{-1}B \Rightarrow A = Y^{-1}B$

$\therefore \quad A = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1-\lambda \\ 7\lambda - 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 - 2\lambda + 7\lambda - 2 \\ -3 + 3\lambda + 7\lambda - 2 \end{pmatrix}$

$\therefore \quad A = \frac{1}{5}\begin{pmatrix} 5\lambda \\ 10\lambda - 5 \end{pmatrix} \Rightarrow A = \begin{pmatrix} \lambda \\ 2\lambda - 1 \end{pmatrix} \quad A(\lambda, 2\lambda - 1)$

7. The rectangular hyperbola, H, has cartesian equation $xy = 25$

The point $P\left(5p, \dfrac{5}{p}\right)$, and the point $Q\left(5q, \dfrac{5}{q}\right)$, where $p, q \neq 0, p \neq q$, are points on the rectangular hyperbola H.

(a) Show that the equation of the tangent at point P is

$$p^2y + x = 10p$$

(4)

(b) Write down the equation of the tangent at point Q.

(1)

The tangents at P and Q meet at the point N.

Given $p + q \neq 0$,

(c) show that point N has coordinates $\left(\dfrac{10pq}{p+q}, \dfrac{10}{p+q}\right)$.

(4)

The line joining N to the origin is perpendicular to the line PQ.

(d) Find the value of p^2q^2.

(5)

a) $y = 25x^{-1} \Rightarrow \dfrac{dy}{dx} = -25x^{-2} = \dfrac{-25}{x^2}$ $\quad x = 5p \quad \therefore \dfrac{dy}{dx} = \dfrac{-1}{p^2}$

$y - y_1 = m(x - x_1) \Rightarrow y - \frac{5}{p} = \frac{-1}{p^2}(x - 5p)$

$\Rightarrow p^2y - 5p = -x + 5p \Rightarrow p^2y + x = 10p$ #

b) $q^2y + x = 10q$

c) $x = 10p - p^2y = 10q - q^2y$

$\Rightarrow 10(p-q) = (p^2 - q^2)y$

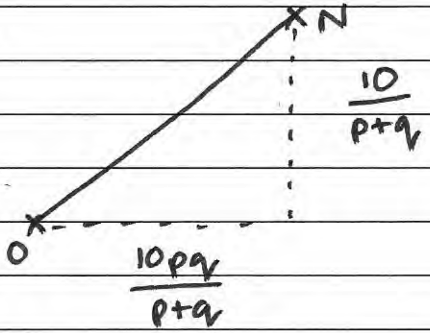
$\Rightarrow 10\cancel{(p-q)} = (p+q)\cancel{(p-q)}y \quad \therefore y = \dfrac{10}{p+q}$

$$x = 10p - p^2 y = 10p - p^2\left(\frac{10}{p+q}\right)$$

$$x = \frac{10p(p+q) - 10p^2}{p+q} = \frac{10p^2 + 10pq - 10p^2}{p+q}.$$

$$\therefore\ x = \frac{10pq}{p+q} \qquad \therefore\ N\left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$$

d)



$$\therefore\ M_{ON} = \frac{\frac{10}{\cancel{p+q}}}{\frac{10pq}{\cancel{p+q}}} = \frac{10}{10pq} = \frac{1}{pq}$$

$$\therefore\ M_{PQ} = -pq.$$

$$P\left(sp, \frac{s}{p}\right) \quad Q\left(sq, \frac{s}{q}\right) \qquad M_{PQ} = \frac{\frac{s}{q} - \frac{s}{p}}{sq - sp} = \frac{\frac{sp - \cancel{sq}}{pq}}{-(sp - \cancel{sq})}$$

$$\Rightarrow M_{PQ} = \frac{-1}{pq} \qquad \therefore\ -pq = -\frac{1}{pq} \Rightarrow -p^2q^2 = -1$$

$$\therefore\ p^2q^2 = 1$$

8. (a) Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r(r+3) = \tfrac{1}{3}n(n+1)(n+5)$$

(6)

(b) A sequence of positive integers is defined by

$$u_1 = 1,$$
$$u_{n+1} = u_n + n(3n+1), \qquad n \in \mathbb{Z}^+$$

Prove by induction that

$$u_n = n^2(n-1)+1, \qquad n \in \mathbb{Z}^+$$

(5)

$n=1$ $\quad \sum_{r=1}^{1} r(r+3) = 1(4) = 4 \qquad \frac{1}{3}n(n+1)(n+5) = \frac{1}{3}(1)(2)(6) = 4$

$\therefore$ true for $n=1$

$n=2$ $\quad \sum_{r=1}^{2} r(r+3) = 2(5)+4 = 14 \qquad \frac{1}{3}n(n+1)(n+5) = \frac{1}{3}(2)(3)(7) = 14$

$\therefore$ true for $n=2$

assume true for $n=k$ $\quad \therefore \quad \sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5)$

$n=k+1$

$$\sum_{r=1}^{k+1} r(r+3) = \tfrac{1}{3}(k+1)(k+2)(k+6)$$

$$\sum_{1}^{k+1} r(r+3) = \sum_{k+1}^{k} r(r+3) + (k+1)(k+4)$$

$$= \tfrac{1}{3}k(k+1)(k+5) + (k+1)(k+4) = \tfrac{1}{3}(k+1)\left[k(k+5)+3(k+4)\right]$$

$$= \tfrac{1}{3}(k+1)\left[k^2+8k+12\right] = \tfrac{1}{3}(k+1)(k+2)(k+6) \quad \#$$

true for $n=1, n=2$. If true for $n=k$, then true for $n=k+1$.
Therefore, by induction true for all $n \in \mathbb{Z}^+$

b) $u_1 = 1$

$(n=1)\ u_2 = 1 + 1(4) = 5$

$(n=2)\ u_3 = 5 + 2(7) = 19$

$u_1 = 1^2(1-1) + 1 = 1$

$u_2 = 2^2(2-1) + 1 = 5 \Rightarrow$ true for $n=1$

$u_3 = 3^2(3-1) + 1 = 19 \Rightarrow$ true for $n=2$

assume true for $n=k$ so $u_{k+1} = u_k + k(3k+1)$

$u_k = k^2(k-1) + 1 \Rightarrow u_{k+1} = (k+1)^2(k+1-1) + 1 = k(k+1)^2 + 1$

$n = k+1$

$$u_{k+1} = u_k + k(3k+1)$$
$$= (k^2(k-1) + 1) + k(3k+1)$$
$$= k^3 - k^2 + 1 + 3k^2 + k$$
$$= k^3 + 2k^2 + k + 1$$
$$= k(k^2 + 2k + 1) + 1$$
$$= k(k+1)^2 + 1 \quad \#$$

$\therefore$ true for $n = 1, 2$

if true for $n=k$, then true for $n=k+1$

$\therefore$ by induction, true for all $n \in \mathbb{Z}^+$

9.

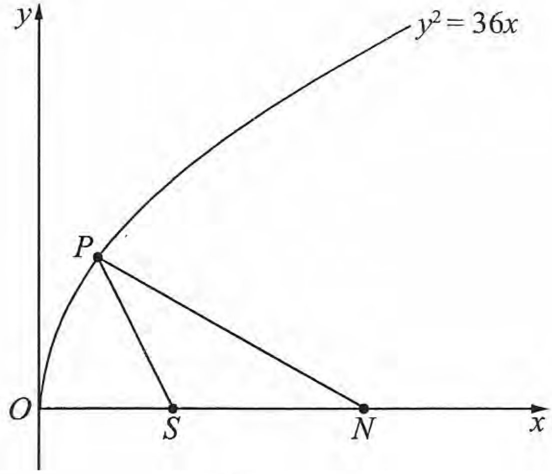


Figure 1

Figure 1 shows a sketch of part of the parabola with equation $y^2 = 36x$.

The point P (4, 12) lies on the parabola.

(a) Find an equation for the normal to the parabola at P.

(5)

This normal meets the x-axis at the point N and S is the focus of the parabola, as shown in Figure 1.

(b) Find the area of triangle PSN.

(4)

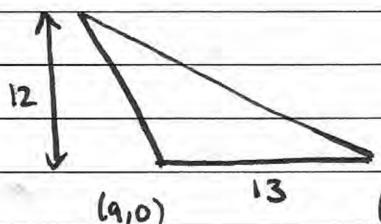
$y^2 = 36x \qquad y^2 = 4ax \therefore a = 9 \qquad S(9,0)$

$\frac{d}{dx}y^2 = \frac{d}{dx}36x \Rightarrow 2y\frac{dy}{dx} = 36 \therefore \frac{dy}{dx} = \frac{18}{y}$

$y = 12 \Rightarrow M_t = \frac{18}{12} = \frac{3}{2} \Rightarrow M_n = -\frac{2}{3}$

$y - 12 = -\frac{2}{3}(x - 4)$

b) $y = 0 \Rightarrow \quad -36 = -2x + 8 \Rightarrow 2x = 44 \Rightarrow x = 22$



Area $= \frac{1}{2}bh = \frac{1}{2} \times 13 \times 12 = \underline{78}$