

1. Given that $z_1 = 1 - i$,

(a) find $\arg(z_1)$.

(2)

Given also that $z_2 = 3 + 4i$, find, in the form $a + ib$, $a, b \in \mathbb{R}$,

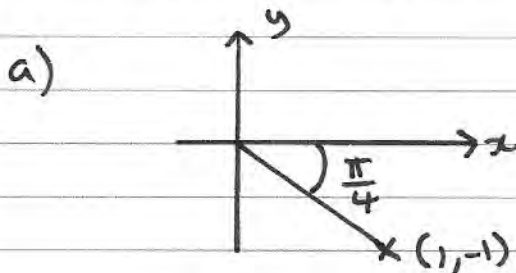
(b) $z_1 z_2$,

(2)

(c) $\frac{z_2}{z_1}$.

(3)

In part (b) and part (c) you must show all your working clearly.



$$\therefore \arg(z_1) = -\frac{\pi}{4}$$

$$\begin{aligned} \text{b) } z_1 z_2 &= (1 - i)(3 + 4i) = 3 + 4i - 3i - 4i^2 \\ &= \underline{7 + i} \end{aligned}$$

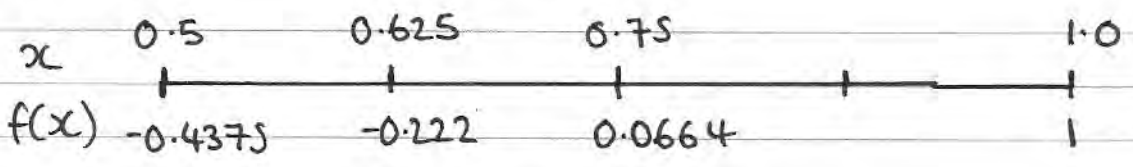
$$\text{c) } \frac{z_2}{z_1} = \frac{3 + 4i}{1 - i} \frac{(1 + i)}{(1 + i)} = \frac{3 + 4i + 3i + 4i^2}{1 - i^2}$$

$$= \frac{-1 + 7i}{2} = -\frac{1}{2} + \frac{7}{2}i$$

2. (a) Show that $f(x) = x^4 + x - 1$ has a real root α in the interval $[0.5, 1.0]$. (2)
- (b) Starting with the interval $[0.5, 1.0]$, use interval bisection twice to find an interval of width 0.125 which contains α . (3)
- (c) Taking 0.75 as a first approximation, apply the Newton Raphson process twice to $f(x)$ to obtain an approximate value of α . Give your answer to 3 decimal places. (5)

a) $f(0.5) = -0.4375$
 $f(1.0) = 1$ } \therefore sign change
 $\alpha \in [0.5, 1.0]$

b)



$\therefore \alpha \in [0.625, 0.750]$

c) $f'(x) = 4x^3 + 1$

$x_0 = 0.75$

$x_1 = 0.75 - \frac{f(0.75)}{f'(0.75)} = 0.7252906977$

$x_2 = 0.725... - \frac{f(0.725...)}{f'(0.725...)} = \underline{0.724 (3dp)}$

3. A parabola C has cartesian equation $y^2 = 16x$. The point $P(4t^2, 8t)$ is a general point on C .

(a) Write down the coordinates of the focus F and the equation of the directrix of C . (3)

(b) Show that the equation of the normal to C at P is $y + tx = 8t + 4t^3$. (5)

$$a) y^2 = 16x = 4ax \Rightarrow a = 4$$

$$\text{focus } (4, 0) \quad \text{directrix } x = -4$$

$$b) \frac{d}{dx} y^2 = \frac{d}{dx} 16x \Rightarrow 2y \frac{dy}{dx} = 16$$

$$\Rightarrow \frac{dy}{dx} = \frac{16}{2y} = \frac{8}{y} \quad \text{Mt at } P = \frac{8}{8t} = \frac{1}{t}$$

$$\therefore M_n \text{ at } P = -t$$

$$y - y_1 = m(x - x_1)$$

$$y - 8t = -t(x - 4t^2)$$

$$y - 8t = -tx + 4t^3$$

$$\therefore y + tx = 8t + 4t^3 \quad \#$$

4. A right angled triangle T has vertices $A(1,1)$, $B(2,1)$ and $C(2,4)$. When T is transformed

by the matrix $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the image is T' .

(a) Find the coordinates of the vertices of T' .

(2)

(b) Describe fully the transformation represented by \mathbf{P} .

(2)

The matrices $\mathbf{Q} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ represent two transformations. When T is transformed by the matrix \mathbf{QR} , the image is T'' .

(c) Find \mathbf{QR} .

(2)

(d) Find the determinant of \mathbf{QR} .

(2)

(e) Using your answer to part (d), find the area of T'' .

(3)

$$a) T' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \end{pmatrix}$$

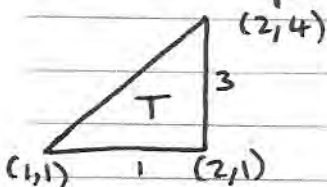
$$A'(1,1) \quad B'(1,2) \quad C'(4,2)$$

b) reflection through $y=x$

$$c) \mathbf{QR} = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$d) \det \mathbf{QR} = (-2)(2) - (0)(0) = -4$$

$$e) \text{Area of } T'' = 4 \times \text{Area of } T$$



$$\text{Area of } T = \frac{3}{2} \quad \therefore \text{Area of } T'' = 6$$

5. The roots of the equation

$$z^3 - 8z^2 + 22z - 20 = 0$$

are z_1, z_2 and z_3 .

(a) Given that $z_1 = 3+i$, find z_2 and z_3 .

(4)

(b) Show, on a single Argand diagram, the points representing z_1, z_2 and z_3 .

(2)

$$a) z_1 = 3+i \Rightarrow z_2 = 3-i$$

$$\text{Sum of roots} = 6$$

$$\text{product of roots} = (3+i)(3-i) = 10$$

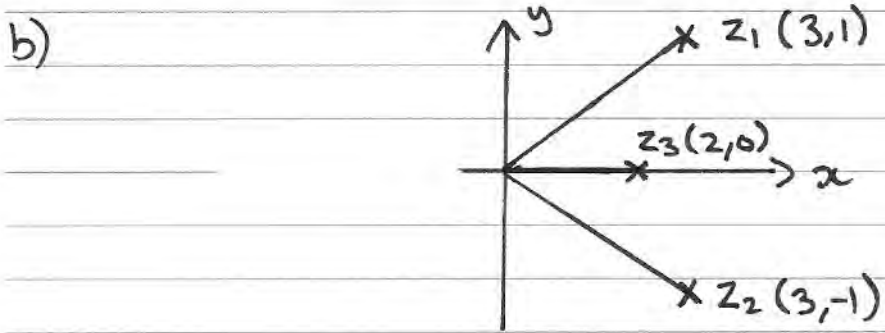
$$\therefore x^2 - (z_1+z_2)x + (z_1z_2) = 0$$

$$\Rightarrow x^2 - 6x + 10$$

$$(z^2 - 6z + 10)(az + b)$$

$$\therefore a=1 \quad b=-2 \quad (z^2 - 6z + 10)(z-2) = 0$$

$$z_1 = 3+i \quad z_2 = 3-i \quad z_3 = 2$$



6. (a) Prove by induction

$$\sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2 \quad (5)$$

(b) Using the result in part (a), show that

$$\sum_{r=1}^n (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8) \quad (3)$$

(c) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$. (3)

$$n=1 \quad \sum_{r=1}^1 r^3 = 1^3 = 1$$

$$\frac{1}{4}n^2(n+1)^2 = \frac{1}{4}(1)^2(2)^2 = 1$$

\therefore true for $n=1$

assume true for $n=k$

$$\sum_{r=1}^k r^3 = \frac{1}{4}k^2(k+1)^2$$

$$\begin{aligned} n=k+1 \quad \sum_{r=1}^{k+1} r^3 &= \frac{1}{4}(k+1)^2(k+1+1)^2 \\ &= \frac{1}{4}(k+1)^2(k+2)^2 \end{aligned}$$

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^k r^3 + (k+1)^3$$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= (k+1)^2 \left[\frac{1}{4}k^2 + (k+1) \right]$$

$$= \frac{1}{4}(u+1)^2 [u^2 + 4u + 4]$$

$$= \frac{1}{4}(u+1)^2 (u+2)^2 \quad \#$$

\therefore true if $n=1$

true for $n=u+1$, if true for $n=u$

\therefore true for all $n \in \mathbb{Z}$ by induction.

$$b) \sum_{r=1}^n r^3 - 2 = \sum_{r=1}^n r^3 - \sum_{r=1}^n 2$$

$$= \frac{1}{4}n^2(n+1)^2 - 2n$$

$$= \frac{1}{4}n [n(n+1)^2 - 8]$$

$$= \frac{1}{4}n [n(n^2 + 2n + 1) - 8]$$

$$= \frac{1}{4}n [n^3 + 2n^2 + n - 8] \quad \#$$

$$c) \sum_{r=20}^{50} (r^3 - 2) = \sum_{r=1}^{50} (r^3 - 2) - \sum_{r=1}^{19} (r^3 - 2)$$

$$= \frac{1}{4}(50) [50^3 + 2(50)^2 + (50) - 8]$$

$$- \frac{1}{4}(19) [19^3 + 2(19)^2 + (19) - 8]$$

$$- \begin{array}{r} 1625525 \\ - 36062 \\ \hline \end{array}$$

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7. A sequence can be described by the recurrence formula

$$u_{n+1} = 2u_n + 1, \quad n \geq 1, \quad u_1 = 1$$

(a) Find u_2 and u_3 .

(2)

(b) Prove by induction that $u_n = 2^n - 1$

(5)

$$a) \quad u_1 = 1$$

$$u_2 = 2u_1 + 1 = 2 \times 1 + 1 = 3$$

$$u_3 = 2u_2 + 1 = 2 \times 3 + 1 = 7$$

$$b) \quad n=1 \quad u_1 = 1 \quad u_1 = 2^1 - 1 = 1 \quad \checkmark$$

$$n=2 \quad u_2 = 3 \quad u_2 = 2^2 - 1 = 3 \quad \checkmark$$

$$n=3 \quad u_3 = 7 \quad u_3 = 2^3 - 1 = 7 \quad \checkmark$$

\therefore true for $n=1, n=2, n=3$

assume true for $n=k \therefore 2^k - 1 = u_k$

$$n=k+1 \quad u_{k+1} = 2^{k+1} - 1$$

$$u_{k+1} = 2u_k + 1 = 2[2^k - 1] + 1$$

$$= 2^{k+1} - 2 + 1$$

$$= 2^{k+1} - 1 \quad \#$$

\therefore true for $n=1, n=2$

if true for $n=k$, true for $n=k+1$

\therefore by induction true for all $n \in \mathbb{Z}$

8.

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$$

(a) Show that A is non-singular.

(2)

(b) Find B such that $BA^2 = A$.

(4)

$$a) \det A = 0 \times 3 - 1 \times 2 = -2$$

$\det A \neq 0 \Rightarrow A$ is non-singular

$$b) BA^2 = A \Rightarrow BA^2A^{-1} = AA^{-1} = I$$

$$\Rightarrow BA^2A^{-1}A^{-1} = A^{-1}$$

$$\therefore B = A^{-1}$$

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 3 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

9. The rectangular hyperbola H has cartesian equation $xy = 9$

The points $P\left(3p, \frac{3}{p}\right)$ and $Q\left(3q, \frac{3}{q}\right)$ lie on H , where $p \neq \pm q$.

(a) Show that the equation of the tangent at P is $x + p^2y = 6p$.

(4)

(b) Write down the equation of the tangent at Q .

(1)

The tangent at the point P and the tangent at the point Q intersect at R .

(c) Find, as single fractions in their simplest form, the coordinates of R in terms of p and q .

(4)

$$xy = 9 = c^2 \quad \therefore c^2 = 9$$

$$y = \frac{9}{x} \Rightarrow y = 9x^{-1}$$

$$\therefore \frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$$

$$\Rightarrow M_t \text{ at } P = \frac{-9}{(3p)^2} = -\frac{1}{p^2} \quad P\left(3p, \frac{3}{p}\right)$$

$$y - y_1 = m(x - x_1) \Rightarrow y - \frac{3}{p} = -\frac{1}{p^2}(x - 3p)$$

$$\textcircled{\times p^2} \Rightarrow p^2y - 3p = -x + 3p$$

$$\therefore x + p^2y = 6p \quad \#$$

$$b) \quad x + q^2y = 6q$$

$$c) \quad x = 6p - p^2y \quad ; \quad x = 6q - q^2y$$

$$\text{at } R \Rightarrow 6p - p^2y = 6q - q^2y$$

$$q^2y - p^2y = 6q - 6p$$

$$\therefore y(q^2 - p^2) = 6q - 6p$$

$$\Rightarrow y = \frac{6(q-p)}{(q-p)(q+p)} = \frac{6}{p+q}$$

$$x = 6p - p^2 y = 6p - p^2 \left(\frac{6}{p+q} \right)$$

$$x = \frac{6p(p+q) - 6p^2}{p+q} = \frac{\cancel{6p^2} + 6pq - \cancel{6p^2}}{p+q}$$

$$x = \frac{6pq}{p+q} \quad R \left(\frac{6pq}{p+q}, \frac{6}{p+q} \right)$$