1. The table below shows the least costs, in pounds, of travelling between six cities, A, B, C, D, E and F.

	A	B	C	D	E	F
A	–	36	18	28	24	22
B	36	–	54	22	20	27
C	18	54	–	42	27	24
D	28	22	42	–	20	30
E	24	20	27	20	–	13
F	22	27	24	30	13	–

Vicky must visit each city at least once. She will start and finish at A and wishes to minimise the total cost.

(a) Use Prim's algorithm, starting at A, to find a minimum spanning tree for this network.
(2)

(b) Use your answer to part (a) to help you calculate an initial upper bound for the length of Vicky's route.
(1)

(c) Show that there are two nearest neighbour routes that start from A. You must make your routes and their lengths clear.
(3)

(d) State the best upper bound from your answers to (b) and (c).
(1)

(e) Starting by deleting A, and all of its arcs, find a lower bound for the route length.
(4)
(Total 11 marks)

2. (a) Explain the difference between the classical and the practical travelling salesperson problems.
(2)

The table below shows the distances, in km, between six data collection points, A, B, C, D, E, and F.

	A	**B**	**C**	**D**	**E**	**F**
A	-	77	34	56	67	21
B	77	-	58	58	36	74
C	34	58	-	73	70	42
D	56	58	73	-	68	38
E	67	36	70	68	-	71
F	21	74	42	38	71	-

Rachel must visit each collection point. She will start and finish at A and wishes to minimise the total distance travelled.

(b) Starting at A, use the nearest neighbour algorithm to obtain an upper bound. Make your method clear.

(3)

Starting at B, a second upper bound of 293 km was found.

(c) State the better upper bound of these two, giving a reason for your answer.

(1)

By deleting A, a lower bound was found to be 245 km.

(d) By deleting B, find a second lower bound. Make your method clear.

(4)

(e) State the better lower bound of these two, giving a reason for your answer.

(1)

(f) Taking your answers to (c) and (e), use inequalities to write down an interval that must contain the length of Rachel's optimal route.

(1)
(Total 12 marks)

3. Explain what is meant, in a network, by

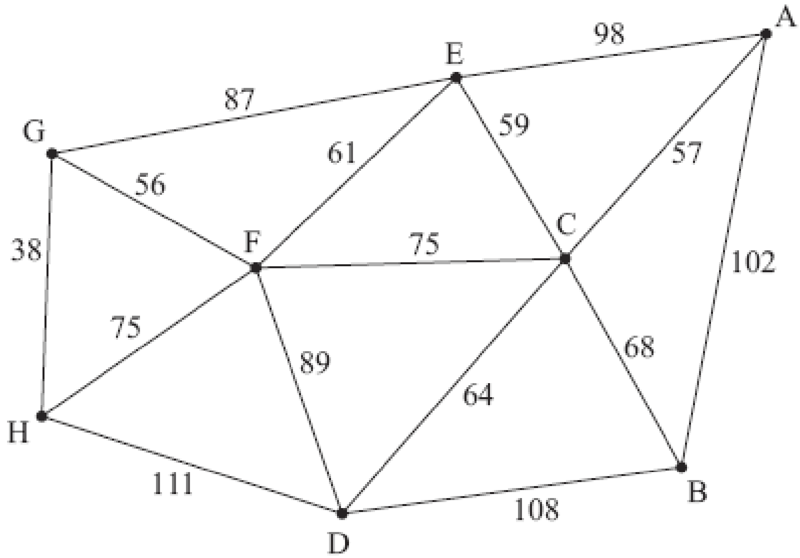
(a) a walk,

(2)

(b) a tour.

(2)
(Total 4 marks)

4.



The network in the diagram above shows the distances, in km, between eight weather data collection points. Starting and finishing at A, Alice needs to visit each collection point at least once, in a minimum distance.

(a) Obtain a minimum spanning tree for the network using Kruskal's algorithm, stating the order in which you select the arcs.

(2)

(b) Use your answer to part (a) to determine an initial upper bound for the length of the route.

(1)

(c) Starting from your initial upper bound use short cuts to find an upper bound, which is below 630km. State the corresponding route.

(4)

(d) Use the nearest neighbour algorithm starting at B to find a second upper bound for the length of the route.

(3)

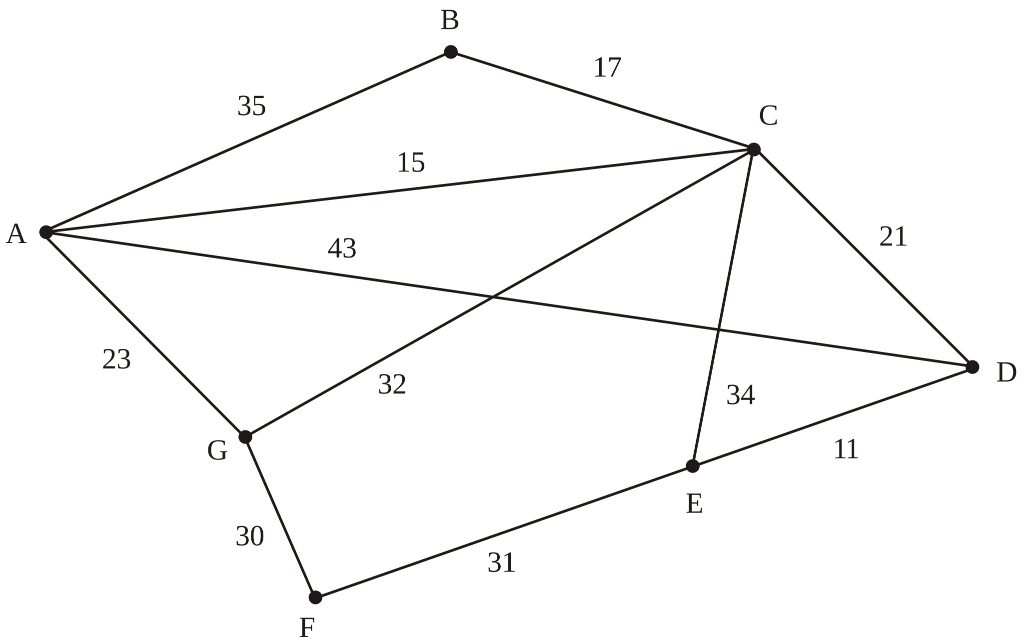
(e) By deleting C, and all of its arcs, find a lower bound for the length of the route.

(4)

(f) Use your results to write down the smallest interval which you are confident contains the optimal length of the route.

(2)
(Total 16 marks)

5.



The network above shows the distances, in miles, between seven gift shops, *A*, *B*, *C*, *D*, *E*, *F* and *G*.

The area manager needs to visit each shop. She will start and finish at shop A and wishes to minimise the total distance travelled.

(a) By inspection, complete the two copies of the table of least distances below.

	A	*B*	*C*	*D*	*E*	*F*	*G*
A	–		15	36		53	23
B		–	17	38	49	80	49
C	15	17	–	21		62	32
D	36	38	21	–	11	42	
E		49		11	–	31	61
F	53	80	62	42	31	–	30
G	23	49	32		61	30	–

	A	*B*	*C*	*D*	*E*	*F*	*G*
A	–		15	36		53	23
B		–	17	38	49	80	49
C	15	17	–	21		62	32
D	36	38	21	–	11	42	
E		49		11	–	31	61
F	53	80	62	42	31	–	30
G	23	49	32		61	30	–

(4)

(b) Starting at A, and making your method clear, find an upper bound for the route length, using the nearest neighbour algorithm.

(3)

(c) By deleting A, and all of its arcs, find a lower bound for the route length.

(4)
(Total 11 marks)

6. A college wants to offer five full-day activities with a different activity each day from Monday to Friday. The sports hall will only be used for these activities. Each evening the caretaker will prepare the hall by putting away the equipment from the previous activity and setting up the hall for the activity next day. On Friday evening he will put away the equipment used that day and set up the hall for the following Monday.

The 5 activities offered are Badminton (B), Cricket nets (C), Dancing (D), Football coaching (F) and Tennis (T). Each will be on the same day from week to week.

The college decides to offer the activities in the order that minimises the total time the caretaker has to spend preparing the hall each week.

The hall is initially set up for Badminton on Monday.

The table below shows the time, in minutes, it will take the caretaker to put away the equipment from one activity and set up the hall for the next.

		To				
	Time	*B*	*C*	*D*	*F*	*T*
From	*B*	–	108	150	64	100
	C	108	–	54	104	60
	D	150	54	–	150	102
	F	64	104	150	–	68
	T	100	60	102	68	–

(a) Explain why this problem is equivalent to the travelling salesman problem.

(2)

A possible ordering of activities is

Monday	Tuesday	Wednesday	Thursday	Friday
B	*C*	*D*	*F*	*T*

(b) Find the total time taken by the caretaker each week using this ordering.

(2)

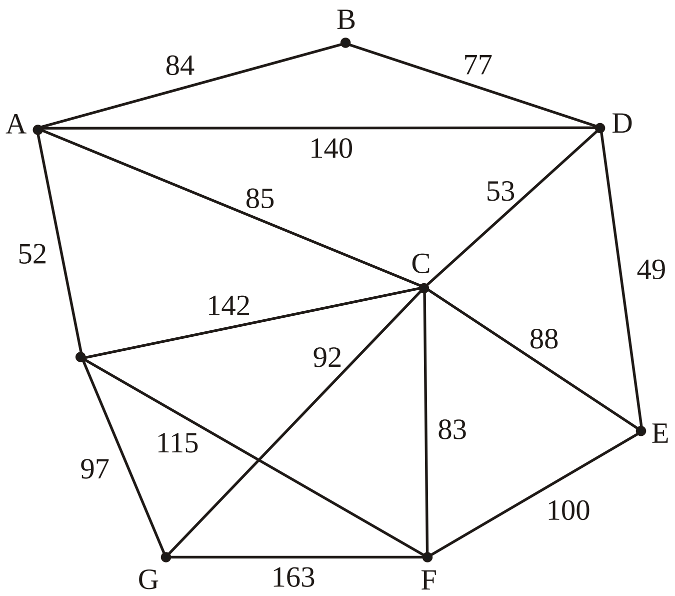
(c) Starting with Badminton on Monday, use a suitable algorithm to find an ordering that reduces the total time spent each week to less than 7 hours.

(3)

(d) By deleting *B*, use a suitable algorithm to find a lower bound for the time taken each week. Make your method clear.

(4)
(Total 11 marks)

7.



The network in the figure above, shows the distances in km, along the roads between eight towns, A, B, C, D, E, F, G and H. Keith has a shop in each town and needs to visit each one. He wishes to travel a minimum distance and his route should start and finish at A.

By deleting D, a lower bound for the length of the route was found to be 586 km.
By deleting F, a lower bound for the length of the route was found to be 590 km.

(a) By deleting C, find another lower bound for the length of the route. State which is the best lower bound of the three, giving a reason for your answer.

(5)

(b) By inspection complete the table of least distances.

Table of least distances

	A	B	C	D	E	F	G	H
A	-	84	85	138	173		149	52
B	84	-	130	77	126	213	222	136

C	85	130	-	53	88	83	92	
D	138	77	53	-	49			190
E	173	126	88	49	-	100	180	215
F		213	83		100	-	163	115
G	149	222	92		180	163	-	97
H	52	136		190	215	115	97	-

(4)

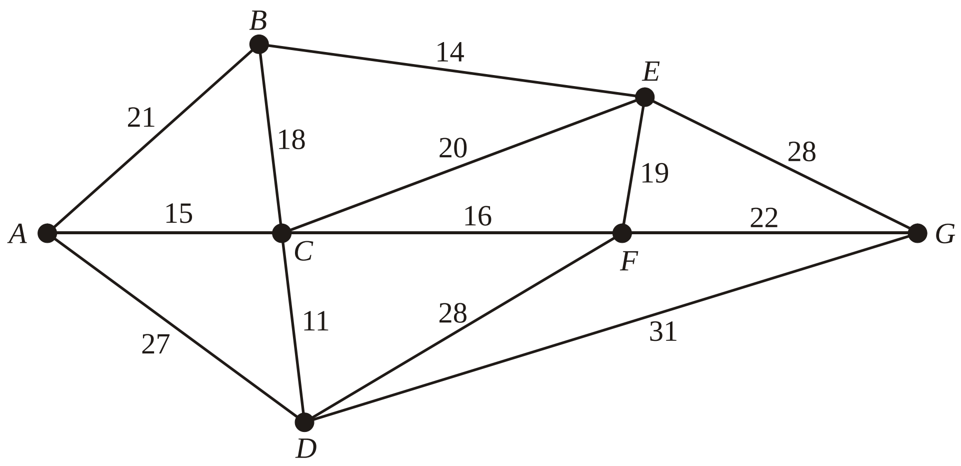
The table can now be taken to represent a complete network.

The nearest neighbour algorithm was used to obtain upper bounds for the length of the route:
Starting at D, an upper bound for the length of the route was found to be 838 km.
Starting at F, an upper bound for the length of the route was found to be 707 km.

(c) Starting at C, use the nearest neighbour algorithm to obtain another upper bound for the length of the route. State which is the best upper bound of the three, giving a reason for your answer.

(4)
(Total 13 marks)

8.



The network in the diagram shows the distances, in km, of the cables between seven electricity relay stations *A*, *B*, *C*, *D*, *E*, *F* and *G*. An inspector needs to visit each relay station. He wishes to travel a minimum distance, and his route must start and finish at the same station.

By deleting *C*, a lower bound for the length of the route is found to be 129 km.

(a) Find another lower bound for the length of the route by deleting *F*. State which is the better lower bound of the two.

(5)

(b) By inspection, complete the table of least distances.

(2)

The table can now be taken to represent a complete network.

(c) Using the nearest-neighbour algorithm, starting at F, obtain an upper bound to the length of the route. State your route.

(4)
(Total 11 marks)

9. The table shows the least distances, in km, between five towns, A, B, C, D and E.

	A	B	C	D	E
A	–	153	98	124	115
B	153	–	74	131	149
C	98	74	–	82	103
D	124	131	82	–	134
E	115	149	103	134	–

Nassim wishes to find an interval which contains the solution to the travelling salesman problem for this network.

(a) Making your method clear, find an initial upper bound starting at A and using

(i) the minimum spanning tree method,

(ii) the nearest neighbour algorithm.

(7)

(b) By deleting E, find a lower bound.

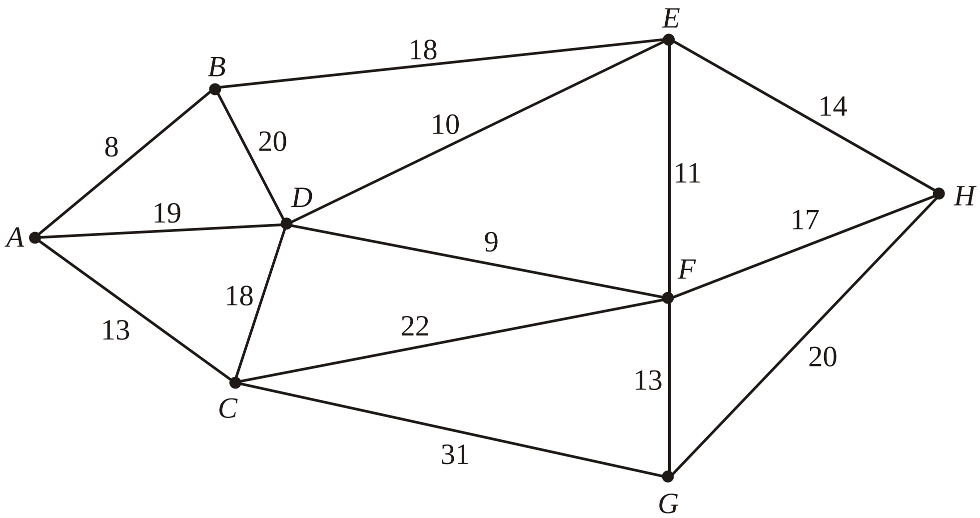
(4)

(c) Using your answers to parts (a) and (b), state the smallest interval that Nassim could correctly write down.

(1)
(Total 12 marks)

10. (a) Explain the difference between the classical and practical travelling salesman problems.

(2)



The network in the diagram above shows the distances, in kilometres, between eight McBurger restaurants. An inspector from head office wishes to visit each restaurant. His route should start and finish at *A*, visit each restaurant at least once and cover a minimum distance.

(b) Obtain a minimum spanning tree for the network using Kruskal's algorithm. You should draw your tree and state the order in which the arcs were added.

(3)

(c) Use your answer to part (b) to determine an initial upper bound for the length of the route.

(2)

(d) Starting from your initial upper bound and using an appropriate method, find an upper bound which is less than 135 km. State your tour.

(3)
(Total 10 marks)

11.

	A	*B*	C	*D*	*E*	*F*	*G*
A	–	165	195	280	130	200	150

B	165	–	90	155	150	235	230
C	195	90	–	170	110	175	190
D	280	155	170	–	150	105	163
E	130	150	110	150	–	90	82
F	200	235	175	105	90	–	63
G	150	230	190	163	82	63	–

An area manager has to visit branches of his company in 7 towns *A*, *B*, *C*, *D*, *E*, *F* and *G*. The table shows the distances, in km, between these 7 towns. The manager lives in town *A* and plans a route starting and finishing at this town. She wishes to visit each town and drive the minimum distance.

(a) Starting from *A*, use Prim's algorithm to find a minimum connector and draw the minimum spanning tree. State the order in which you selected the arcs.

(5)

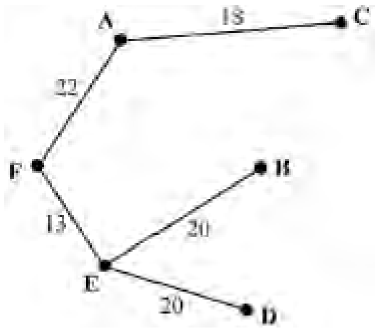
(b) (i) Hence determine an initial upper bound for the length of the route planned by the manager.

(ii) Starting from your initial upper bound and using a short cut, obtain a route with length less than 870 km.

(iii) Find a further cut which produces a route which visits each vertex exactly once and has a length less than 810 km.

(7)
(Total 12 marks)

1. (a)



M1 A1 2

Notes

1M1: Spanning tree found. Allow 1×2×43 across top of table or 93
1A1: CAO must see tree or list of arcs

(b) Minimum Spanning tree length 93, so upper bound is £186 B1ft 1

Note

1B1ft: 186 their ft93 × 2

(c)	A C F E B D A	M1	
	18 24 13 20 22 28 Length 125	A1	
	A C F E D B A		
	18 24 13 20 22 36 Length 133	A1	3

Notes

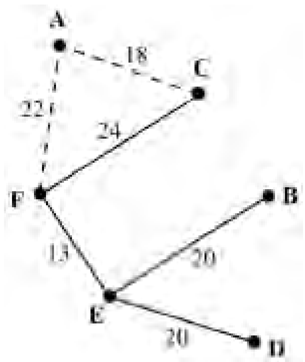
1M1: One Nearest Neighbour each vertex visited at least
once (condone lack of return to start)
1A1: One correct route and length CAO – must return to start.
2A1: Second correct route and length CAO – must return to start.

(d) Best upper bound is £125 B1ft 1

Note

1B1ft: ft but only on three different values.

(e) Delete A



M1 A1

RMST weight = 77
Lower bound = 77 + 18 + 22 = £117 M1 A1 4

Notes

1M1: Finding correct RMST (maybe implicit) 77 sufficient,
or correct numbers. 4 arcs.
1A1: CAO tree or 77.
2M1: Adding 2 least arcs to A, 18 and 22 or 40 only
2A1: CAO 117

[11]

2. (a) In the classical problem each vertex must be visited **only** once.

In the practical problem each vertex must be visited **at least** once. B2 1 0 2

Note

1B1: Generous, on the right lines
bod gets B1

2B1: cao, clear answer.

(b) AFDBECA{1 4 6 3 5 2 } M1 A1

21 + 38 + 58 + 36 + 70 + 34 = 257 A1 3

Note

1M1: Nearest Neighbour each vertex visited once (condone lack of return to start)

1A1: Correct route cao – must return to start.

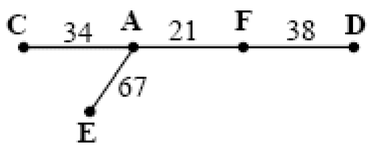
2A1: 257 cao

(c) 257 is the better upper bound, it is lower. B1ft 1

Note

1B1ft: ft their lowest.

(d) R.M.S.T.



M1 A1

Lower bound is 160 + 36 + 58 = 254 M1A1 4

Note

1M1: Finding correct RMST (maybe implicit)
160 sufficient

1A1: cao tree or 160.

2M1: Adding 2 least arcs to B, 36 and 58 only

2A1: 254

	(e)	Better lower bound is 254, it is higher **Note** 1B1ft: ft their highest	B1ft	
	(f)	254<optimal≤257 **Note** 1B1: cao	B1	2

[12]

3.	(a)	A walk is a **finite sequence of arcs** such that the **end vertex of one arc is the start vertex of the next**. 1B1: Probably one of the two below but accept correct relevant statement– bod gets B1, generous. 2B1: A good clear complete answer: End vertex = start vertex + finite.	B2,1,0	2
	(b)	A tour is a walk that visits **every vertex, returning to its stating vertex**. 1B1: Probably one of the two below but accept correct relevant statement– bod gets B1, generous. 2B1: A good clear complete answer: Every vertex + return to start.	B2,1,0	2

<u>From the D1 and D2 glossaries</u>

<u>D1</u>

A **path** is a finite sequence of edges, such that the end vertex of one edge in the sequence is the start vertex of the next, <u>and in which no vertex appears more than once</u>.

A **cycle (circuit)** is a closed path, ie the end vertex of the last edge is the start vertex of the first edge.

<u>D2</u>

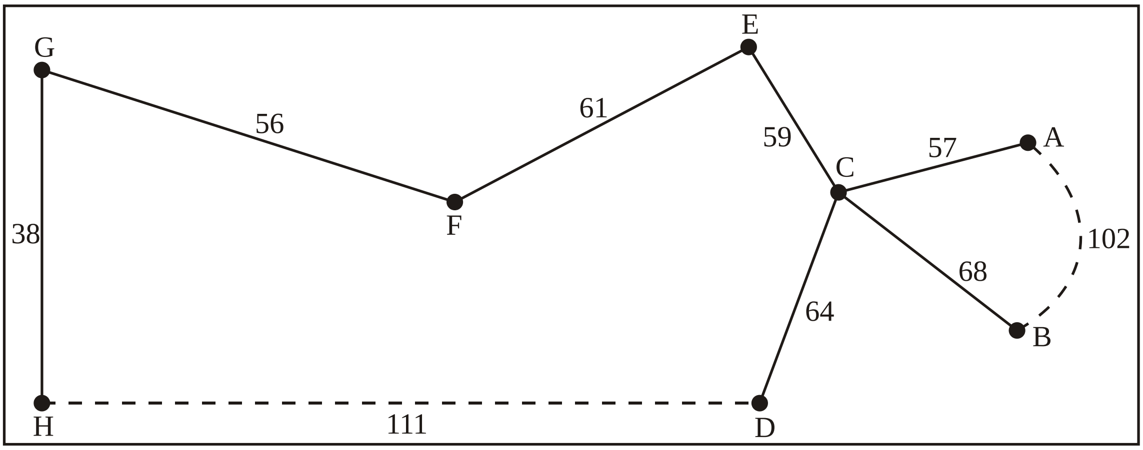
A **walk** in a network is a finite sequence of edges such that the end vertex of one edge is the start vertex of the next.

A walk which visits every vertex, returning to its starting vertex, is called a **tour**.

[4]

4.	(a)	GH(38) GF(56) CA(57) EC(59) FE(61) CD(64) CB(68)	M1A1ft	2
	(b)	$2 \times 403 = 806$ (km)	B1	1

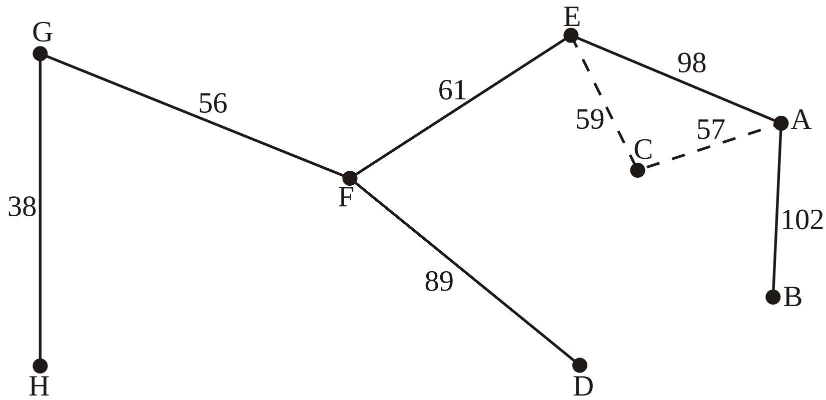
(c)	e.g. DH saves 167	M1A1	
	AB saves 23		
	806 – 190 = 616 (km)	A1	



A1 4

(d)	eg A B C E F G H D C A	M1A1	
	B C A E F G H D B		
	68 + 57 + 98 + 61 + 56 + 38 + 111 + 108 = 597 (km)	A1	3

(e) Delete C



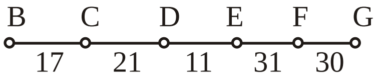
M1A1M1A1ft 4

(f)	RMST weight = 444		
	Lower bound = 444 + 59 + 57 = 560 (km)		
	560 < length ≤ 597	B2,1,0	2

[16]

5.	(a)	Adds 32 to $AB + BA$ (ACB)	B1	
		47 to $AE + EA$ ($ACDE$)	B1	
		32 to $CE + EC$ (CDE)	B1	
		53 to $DG + GD$ (DCG)	B1	4

(b) *A C B D E F G A* M1A1
15 + 17 + 38 + 11 + 31 + 30 + 23 = 165 miles A1 3

(c) e.g. *BC*, *CD*, *DE*, *EF*, *FG*


M1

weight of RSMT = 110 miles A1
Lower bound = 110 + 15 + 23 M1
= 148 miles A1ft 4

[11]

6. (a) Each activity must be visited once and then we return to the starting activity, this must be done in a minimum time B2, 1, 0 2

B2 cao – all 3 bits in the context
B1 cloze 'Bod' is B1 (e.g. not in contect; just 'each activity once' – but not all 3; …)

(b) 108 + 54 + 150 + 68 + 100 = 480 minutes (= 8 hours) M1 A1 2

M1 (maybe implicit) attempting to add 5 values
A1 cao

(c) Use nearest neighbour B F T C D B M1 A1

64 + 68 + 60 + 54 + 150 = 396 minutes (67 hours) A1 3

M1 each vertex visited once – either NN or 2 x mst-shortcut (BD)
A1 cao incl return to B (BFTCDB)
A1 cao (396)

(d)

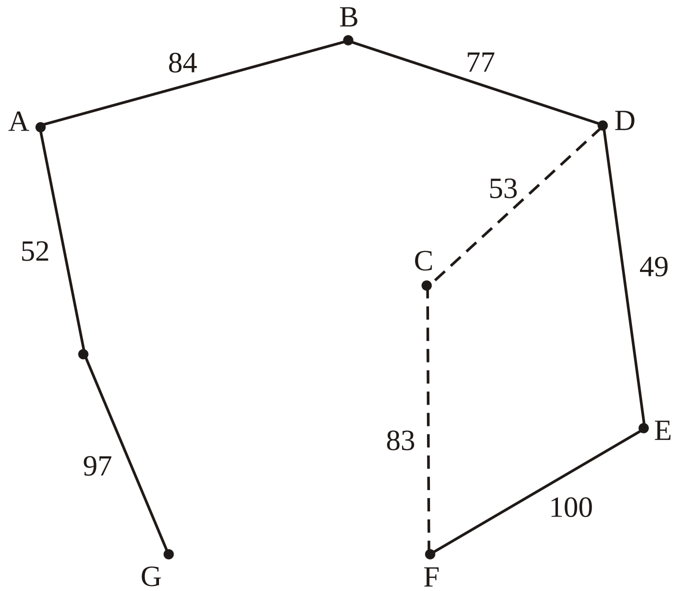

CT, TF, CD (Prim or Kruskal) M1 A1

182 +64 + 100 = 346 minutes M1 A1ft 4

M1 Finding correct minimum spanning tree (maybe implicit) 182 sufficient
A1 cao tree or 182
M1 adding 2 least arcs to B i.e. 100 and 64 only
*A1ft cao ft from their m.s.t. value i.e. 164 and their **tree** length*

[11]

7. (a)



R.M.S.T

e.g. AH, AB, BD, DE	M1	
HG, EF using prim	A1	
length of R M S T = 459		
∴ lower bound = 459 + 53 + 83 = 595 km (deleting c)	A1	
∴ Best lower bound is 595 km, by deleting c	M1 A1ft	5

(b) Adds 167 to AF and FA
137 to CH and HC — B1, 3, 2, 1, 0 — 4
136 to DF and FD
145 to DG and GD

(c)

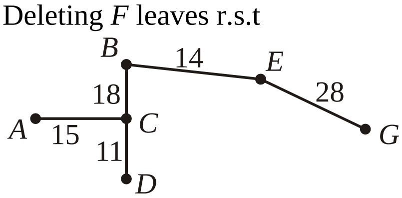
C	D	E	F	H	A	B	G	C
	53	49	120	115	52	84	222	92

M1 A1

∴ Best upper bound is 707 starting at F — B1ft — 4

[13]

8. (a) Deleting *F* leaves r.s.t

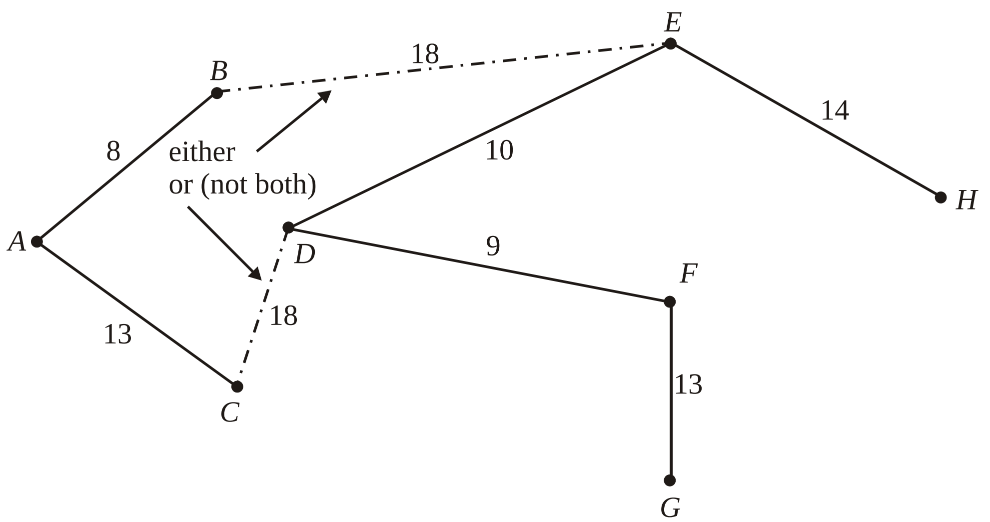


	M1	
r.s.t. length = <u>86</u>	A1	
s_0 lower bound = 86 + 16 + 19 = <u>121</u>	M1 a1	4
∴<u>best L.B is 129</u> by deleting *C*(ft from <u>choice</u>)	B1 ft	1

(b)	Add 33 to *BF* and *FB*	B1		
	Add 31 to *DE* and *ED*	B1	2	
(c)	Tour, visits each vertex, order correct using table of least distances.	M1 A1		
	e.g. *F C D A B E G F* (actual route *F C D C A B E G F*)	A1		
	upper bound of 138 km	A1	4	
				[11]

9.	(a)	(i)	Minimum connector using Prim: *AC*, *CB*, *CD*, *CE*	M1 A1		
			Length = 98 + 74 + 82 + 103 = 357	{1, 3, 2, 4, 5}		
			So upper bound = 2 × 357 = 714	M1 A1	4	
		(ii)	*A* <u>(98)</u> *C* <u>(74)</u> *B* <u>(131)</u> *D* (134) *E* <u>(115)</u> *A*	M1 A1		
			Length = 98 + 74 + 131 + 134 + 115 = 552	A1	3	
	(b)		Residual minimum connector is *AC*, *CB*, *CD*	M1		
			Length 254	A1		
			Lower bound = 254 + 103 + 115 = 472	M1 A1	4	
	(c)		472 ≤ solution ≤ 552	B1 ft	1	
						[12]

10.	(a)	In the *practical* TSP each vertex must be visited *at least once*	B1	
		In the *classical* TSP each vertex must be visited *exactly once*	B1	2
	(b)	AB, DF, DE, (reject EF), $\begin{Bmatrix} FG \\ AC \end{Bmatrix} EH \begin{Bmatrix} DC \\ \text{or} \\ BE \end{Bmatrix}$	M1 A1	



B1 3

(c) Initial upper bound = 2 × 85 = 170 km M1 A1 2

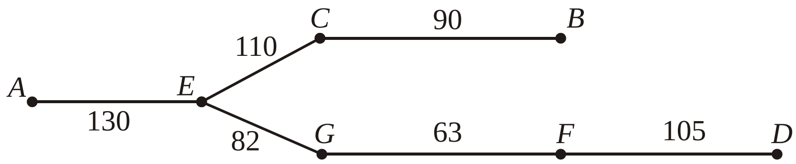
(d) e.g. when *CD* is part of the tree
use *GH* (saving 26) and *BD* (saving 19) giving new u. b.
of 125 km M1
Tour *A B D E H G F D C A* A1 3

(or e.g. when *BE* is part of the tree
use *CG* (saving 40) giving new upper bound of 130 km;
Tour *A B E H E D F G C A*)

[10]

11. (a)

	1	7	6	5	2	4	3
	A	*B*	*C*	*D*	*E*	*F*	*G*
A	–	165	195	280	130	200	150
B	165	–	90	155	150	235	230
C	195	90	–	170	110	175	190
D	280	155	170	–	150	105	163
E	130	150	110	150	–	90	82
F	200	235	175	105	90	–	63
G	150	230	190	163	82	63	–



Order in which arcs are selected: *AE*, *EG*, *GF*, *FD*, *EC*, *CB* M1 A1 5

(b) (i) An initial upper bound is given by 2 (weight of *MST*)

= 2(130 + 110 + 90 + 82 + 63 + 105) = 2 × 580 = 1160 km M1 A1

(ii) Use *BD* instead of *BCEGFD*.

Saving (90 + 110 + 82 + 63 + 105) – 155

= 450 – 155 = 295

Giving upper bound of 1160 – 295 = 865 km

Route is then *AECBDFGEA.* M1 A1

(iii) Using *AG* instead of *AEG* gives a reduction of M1 A1
(130 +82) – 150 = 62 km.

The route is then *AECBDFGA* which visits each vertex once only.

Length is 865 – 62 = 803 km. A1 7

[12]

1. This proved a good first question with over a third of the candidates scoring full marks and over 50% getting at least 10 marks.

In part (a) many candidates did not list the arcs or draw the tree, BD was often included. Almost all were able to correctly find an initial upper bound based on their tree.

In (c) most candidates were able to find the two nearest neighbour routes but a surprisingly large number did not return to A, others found the NN route from A to B and A to D and then, alarmingly, doubled it.

Those who completed (c) correctly usually completed (d) correctly.

Many completed part (e) correctly but BD was, once again, often included in the residual minimum spanning tree.

2. This proved a good discriminator with able candidates producing concise, accurate solutions. Most candidates made a good attempt at the difference between the classical and practical problems, but some muddled the two and others referred to arcs/edges rather than vertices/nodes. Most went on to correctly use the nearest neighbour algorithm but some used it to find a path from A to C and then doubled it. A few candidates found the minimum spanning tree and doubled it. Most candidates selected the lower upper bound. In part (d) a significant number of candidates did not find the correct residual minimum spanning tree, although most did then add the two shortest arcs from B. Most candidates were able to select the higher lower bound.

3. The definition of 'walk' was less successful than the definition of 'tour'. Some candidates wrote the same definition for both parts. Part (a) proved quite challenging with poor use of technical terminology, but most were able to gain credit in (b) with many gaining both marks.

4. Many candidates found the MST correctly but a significant minority used Prim's rather than Kruskal's algorithm. Most were able to use their answer to find a correct intial upper bound. Candidates did not always make their shortcuts clear in part (c). Candidates are reminded that this is a 'methods' paper and they must make their method clear. The most successful candidates were those who stated their short cuts clearly and then listed the route and its length. Some candidates evidently just 'spotted' a route and gained no credit. Most candidates were able to find the NN route in (d), but some did not add the final arc to B, others doubled their route from B to D. In part (e) most candidates found the correct residual minimum spanning tree, although BD was often incorrectly included. The vast majority added in the two leat arcs to C. Full marks in (f) was only possible for those gaining full credit in (c) – (e).

5. Part (a) was poorly done by a surprisingly large number of candidates. The majority of candidates made at least one mistake in calculating the additional distances, with a significant number making 3 or 4 mistakes. Numbers written into the grids were often difficult to read because of subsequent working. Most candidates found the correct NN route and length, although some omitted the return to A. Other candidates found the MST A-G and doubled this to give their upper bound. Part (c) was correctly done by a large majority of candidates. When errors did occur, it was either in finding the wrong tree or adding the incorrect arcs from A.

6. This question proved accessible to all candidates. Most candidates were able to gain some credit in part (a), but few gave complete answers, most were able to relate this practical problem to the TSP, but few made it clear that they must complete each activity once, in a minimum total time and return to the start. The remainder of the question was often very well done. Most candidates obtained the initial upper bound, but a small number omitted the return to B. In part (c) candidates either used Nearest Neighbour or 2 x MST and shortcuts to arrive at a better upper bound. The most common errors were failing to state a route or omitting the final arc to return to B. In part (d) most candidates found the correct lower bound, however the most common error was to select DT (102) in the RMST rather than CT (60).

7. Most were able to make a good attempt at part (a), although some found a **cycle** rather than a minimum spanning **tree**. Some selected the least value as the best lower bound. Candidates were usually able to complete the table correctly but many got 168 instead of 167 for AF and FA, and 142 instead of 137 for CH and HC. Many made a good attempt at part (c) but some did not state the nearest neighbour route and some did not return to C. Most candidates correctly chose their least value as the best upper bound but some selected the greatest.

8. This was well-answered by most candidates. A number of candidates made errors in finding the lower bound, including AB in their RST or using nearest neighbour. Many candidates did not draw the correct conclusion about the better lower bound. Most candidates completed part (b) correctly. In finding the upper bound some candidates did not return form G to F but doubled their route from F to G. Others included the direct route from A to D (27) rather than the least route (26) giving an upper bound of 139km.

9. Most candidates were able to find the upper bounds in part (a) although not all made their method clear (e.g. by listing the order of arc selection) and lost marks accordingly. In part (ii) many did not list a cycle. In part (b) a surprisingly large number of candidates deleted E from their NN cycle rather than their doubled MST. There was also surprisingly frequent misuse of, or incorrect, inequalities, with many candidates not writing a correct mathematical statement.

10. There was a varied response to this question. In part (a) many candidates were unable to explain the difference between the practical and classical problems, often confusing one or the other with route inspection. Most candidates were able to obtain the minimum spanning tree but many used Prim's algorithm instead of Kruskal's. Part (c) was often well done. Part (d) was often well done although some candidates used the nearest neighbour algorithm instead of finding short cuts.

11. No Report available for this question.