02 June 2012

Write your answers in the D2 answer book for this paper.

1. Five workers, A, B, C, D and E, are to be assigned to five tasks, 1, 2, 3, 4 and 5. Each worker is to be assigned to one task and each task must be assigned to one worker.

The cost, in pounds, of assigning each person to each task is shown in the table below. The cost is to be minimised.

	1	2	3	4	5
A	129	127	122	134	135
B	127	125	123	131	132
C	142	131	121	140	139
D	127	127	122	131	136
E	141	134	129	144	143

(a) **Reducing rows first**, use the Hungarian algorithm to obtain an allocation that minimises the cost. You must make your method clear and show the table after each stage.

(8)

(b) Find the minimum cost.

(1)

(Total 9 marks)

1.

	1	2	3	4	5
A	129	127	122	134	135
B	127	125	123	131	132
C	142	131	121	140	139
D	127	127	122	131	136
E	141	134	129	144	143

Reduce rows

	1	2	3	4	5	
A	7	5	0	12	13	-122
B	4	2	0	8	9	-123
C	21	10	0	19	18	-121
D	5	5	0	9	14	-122
E	13	5	0	15	14	-129

reduce Columns

	1	2	3	4	5
A	3	3	0	4	4
B	0	0	0	0	0
C	17	8	0	11	9
D	1	3	0	1	5
E	9	3	0	7	5
	-4	-2	x	-8	-9

2 lines ⇒ not optimal
smallest uncovered = 1

	1	2	3	4	5
A	2	2	0	3	3
B	0	0	1	0	0
C	16	7	0	10	8
D	0	2	0	0	4
E	8	2	0	6	4

3 lines ⇒ not optimal
smallest uncovered = 2

	1	2	3	4	5
A	0	0	0	1	1
B	0	0	3	0	0
C	14	5	0	8	6
D	0	2	2	0	4
E	6	0	0	4	2

5 lines ⇒ optimal

A - 1
B - 5
C - 3
D - 4
E - 2

£647

2. The table shows the least distances, in km, between six towns, A, B, C, D, E and F.

	A	B	C	D	E	F
A	-	16	25	21	12	15
B	16	-	24	22	21	12
C	25	24	-	18	30	27
D	21	22	18	-	15	12
E	12	21	30	15	-	18
F	15	12	27	12	18	-

Toby must visit each town at least once. He will start and finish at A and wishes to minimise the total distance.

(a) Use the nearest neighbour algorithm, starting at A, to find an upper bound for the length of Toby's route.

(3)

(b) Starting by deleting A, and all of its arcs, find a lower bound for the route length.

(4)

(Total 7 marks)

2. (a)

	A	B	C	D	E	F
A	-	16	25	21	12	15
B	16	-	24	22	21	12
C	25	24	-	18	30	27
D	21	22	18	-	15	12
E	12	21	30	15	-	18
F	15	12	27	12	18	-

$A_{12} E_{15} D_{12} F_{12} B_{24} C_{25} A$

AEDFBCA (100) upper bound = 100

(b)

	A	B	C	D	E	F
A	-	16	25	21	12	15
B	16	-	24	22	21	12
C	25	24	-	18	30	27
D	21	22	18	-	15	12
E	12	21	30	15	-	18
F	15	12	27	12	18	-

Prim's : BF(12) ; FD(12); FE(15) ; FD(18)

RMST = 57 + AE(12) + Af(15) = 84

lower bound

3. The table below shows the cost, in pounds, of transporting one tonne of concrete from each of three supply depots, A, B and C, to each of four building sites, D, E, F and G. It also shows the number of tonnes that can be supplied from each depot and the number of tonnes required at each building site. A minimum cost solution is required.

	D	E	F	G	Supply
A	17	19	21	20	18
B	21	20	19	22	23
C	18	17	16	21	29
Demand	15	24	18	13	

The north-west corner method gives the following possible solution.

	D	E	F	G	Supply
A	15	3			18
B		21	2		23
C			16	13	29
Demand	15	24	18	13	

Taking AG as the first entering cell,

(a) use the stepping stone method **twice** to obtain an improved solution. You must make your method clear by stating your shadow costs, improvement indices, routes, entering cells and exiting cells.

(8)

(b) Determine whether your current solution is optimal. Justify your answer.

(4)

(Total 12 marks)

3.

	D	E	F	G	Supply
A	17	19	21	20	18
B	21	20	19	22	23
C	18	17	16	21	29
Demand	15	24	18	13	

	D	E	F	G	Supply
A	15	3 $-\theta$		$+\theta$	18
B		21 $+\theta$	2 $-\theta$		23
C			16 $+\theta$	13 $-\theta$	29
Demand	15	24	18	13	

entering cell AG

$\theta = 2$

exiting cell = BF

	D	E	F	G	Supply
A	15	1		2	18
B		23			23
C			18	11	29
Demand	15	24	18	13	

Improved Solution

		17	19	15	20	
		D	E	F	G	Supply
0	A	17	19	21	20	18
1	B	21	20	19	22	23
1	C	18	17	16	21	29
	Demand	15	24	18	13	

Shadow Costs

	D	E	F	G	Supply
A	X	X	6	X	18
B	3	X	3	1	23
C	0	-3	X	X	29
Demand	15	24	18	13	

Improvement Indices

	D	E	F	G	Supply
A	15	1 $-\theta$		2 $+\theta$	18
B		23			23
C		$+\theta$	18	11 $-\theta$	29
Demand	15	24	18	13	

entering cell = CE

$\theta = 1$

exiting cell = AE

Improved Solution

	D	E	F	G	Supply
A	15			3	18
B		23			23
C		1	18	10	29
Demand	15	24	18	13	

shadow costs

		17	16	15	20	
		D	E	F	G	Supply
0	A	17	19	21	20	18
4	B	21	20	19	22	23
1	C	18	17	16	21	29
	Demand	15	24	18	13	

Improvement Indices

	D	E	F	G	Supply
A	×	3	6	×	18
B	0	×	0	-2	23
C	0	×	×	×	29
Demand	15	24	18	13	

	D	E	F	G	Supply
A					18
B					23
C					29
Demand	15	24	18	13	

The Improved solution is not optimal as there is a negative improvement index.

4. The tableau below is the initial tableau for a maximising linear programming problem in x, y and z which is to be solved.

Basic variable	x	y	z	r	s	t	Value
r	5	$\frac{1}{2}$	0	1	0	0	5
s	1	–2	4	0	1	0	3
t	8	4	6	0	0	1	6
P	–5	–7	–4	0	0	0	0

(a) Starting by increasing y, perform one complete iteration of the simplex algorithm, to obtain tableau T. State the row operations you use.

(5)

(b) Write down the profit equation given by tableau T.

(2)

(c) Use the profit equation from part (b) to explain why tableau T is optimal.

(1)

(Total 8 marks)

4.

b.v.	x	y	z	r	s	t	Value
r	5	$\frac{1}{2}$	0	1	0	0	5
s	1	–2	4	0	1	0	3
t	8	$\boxed{4}$	6	0	0	1	6
P	–5	–7	–4	0	0	0	0

$\theta = 5 \div \frac{1}{2} = 10$

$\theta = 3 \div -2 = -1.5$

$\theta = 6 \div 4 = 1.5^{*}$

(a) *You may not need to use all of these tableaux*

Increase y

b.v.	x	y	z	r	s	t	Value	Row Ops
	5	$\frac{1}{2}$	0	1	0	0	5	R1
	1	-2	4	0	1	0	3	R2
	2	1	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{3}{2}$	R3 ÷ 4
P	-5	-7	-4	0	0	0	0	R4

b.v.	x	y	z	r	s	t	Value	Row Ops
r	4	0	$-\frac{3}{4}$	1	0	$-\frac{1}{8}$	$\frac{17}{4}$	$R1 - \frac{1}{2}R3$
s	5	0	7	0	1	$\frac{1}{2}$	6	R2 + 2R3
y	2	1	$\frac{3}{2}$	0	0	$\frac{1}{4}$	$\frac{3}{2}$	R3
P	9	0	$\frac{13}{2}$	0	0	$\frac{7}{4}$	$\frac{21}{2}$	R4 + 7R3

b) $P + 9x + \frac{13}{2}z + \frac{7}{4}t = \frac{21}{2}$

c) $P = \frac{21}{2} - 9x - \frac{13}{2}z - \frac{7}{4}t$

Since $x, z, t \geq 0$ Increasing x, z or t will reduce the profit.

5. Agent Goodie is planning to break into Evil Doctor Fiendish's secret base.

He uses game theory to determine whether to approach the base from air, sea or land.

Evil Doctor Fiendish decides each day which of three possible plans he should use to protect his base.

Agent Goodie evaluates the situation. He assigns numbers, negative indicating he fails in his mission, positive indicating success, to create a pay-off matrix. The numbers range from –3 (he fails in his mission and is captured) to 5 (he successfully achieves his mission and escapes uninjured) and the pay-off matrix is shown below.

	Fiendish uses plan 1	Fiendish uses plan 2	Fiendish uses plan 3
Air	0	4	5
Sea	2	–3	1
Land	–2	3	–2

(a) Reduce the game so that Agent Goodie has only two choices, explaining your reasoning.

(1)

(b) Use game theory to determine Agent Goodie's best strategy.

(7)

(c) Find the value of the game to Agent Goodie.

(1)

(Total 9 marks)

5.

	Fiendish uses plan 1	Fiendish uses plan 2	Fiendish uses plan 3
Air	0	4	5
Sea	2	–3	1
Land	–2	3	–2

a) Row 1 dominates Row 3 as every value in Row 1 is greater than the corresponding value in Row 3 $\therefore$ delete row 3 from the game.

b)

	F1	F2	F3	Row min
A	0	4	5	0
S	2	-3	1	-3
Column max	2	4	5	

Column minimax (2) $\neq$ Row maximin
$\therefore$ no stable solution exists
$\therefore$ Goodie should play an optimal mixed strategy

<u>Quick method</u> Column 2 dominates Column 3 as every value in Column 2 is smaller than the corresponding value in Column 3
$\therefore$ Column 3 can be deleted from the game

	F1	F2
A	0	4
S	2	-3

let G play A prob $= p$
let G plays S prob $= 1-p$

If f plays 1 $V(G) = 0 + 2(1-p) = 2-2p$
If f plays 2 $V(G) = 4 - 3(1-p) = -3+7p$

$2-2p = -3+7p \Rightarrow 9p = 5 \Rightarrow p = \frac{5}{9}$

$V(G) = 7\left(\frac{5}{9}\right) - 3 = \frac{8}{9}$

Goodie Air prob $= \frac{5}{9}$
Goodie Sea Prob $= \frac{4}{9}$
Goodie Land Prob $= 0$

6.

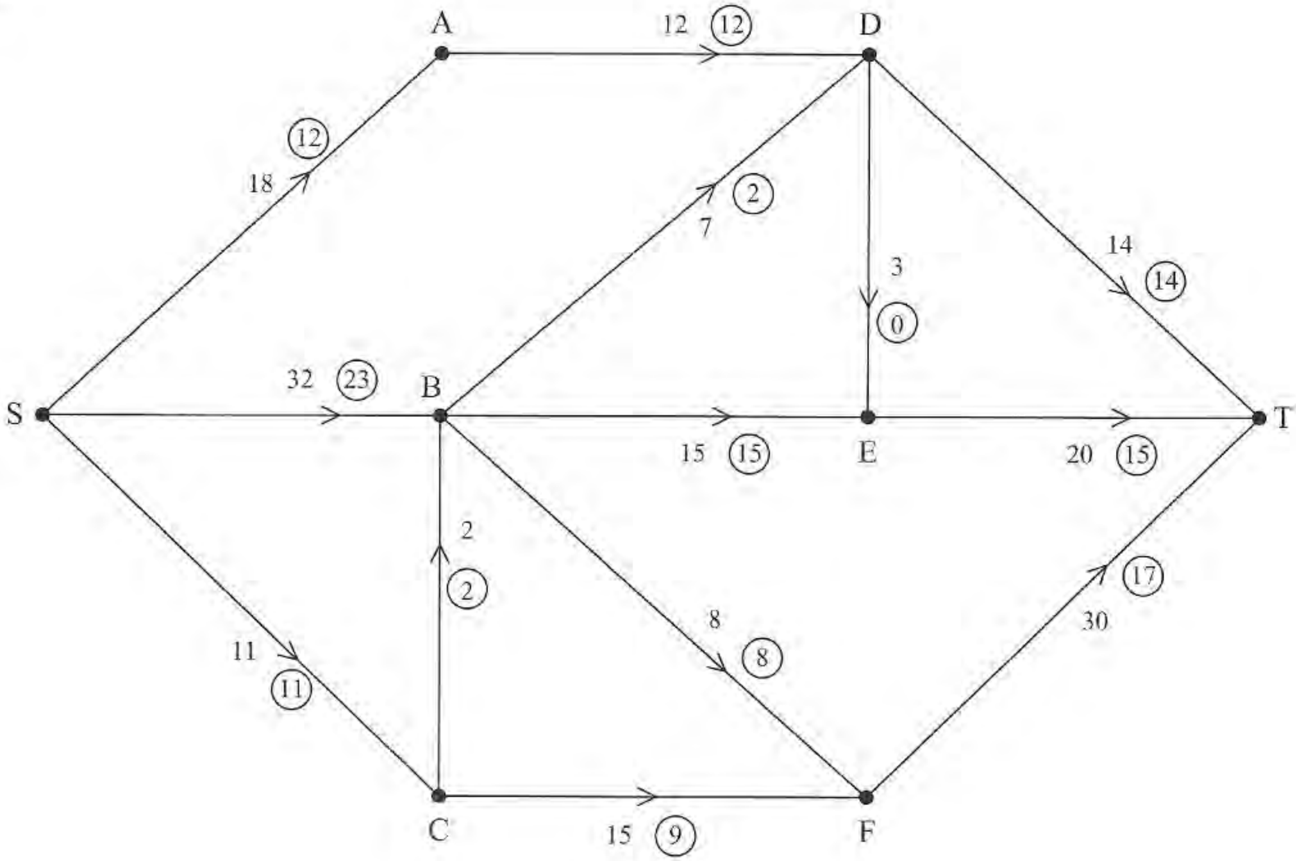

Figure 1

Figure 1 shows a capacitated, directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow.

(a) State the value of the initial flow.

(1)

(b) Complete the initialisation of the labelling procedure on Diagram 1 in the answer book by entering values along SB, BD, CF and FT.

(2)

(c) Hence use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use, together with its flow.

(4)

(d) Draw a maximal flow pattern on Diagram 2 in your answer book.

(2)

(e) Prove that your flow is maximal.

(2)

(Total 11 marks)

6. (a) Value of initial flow 46

(b)

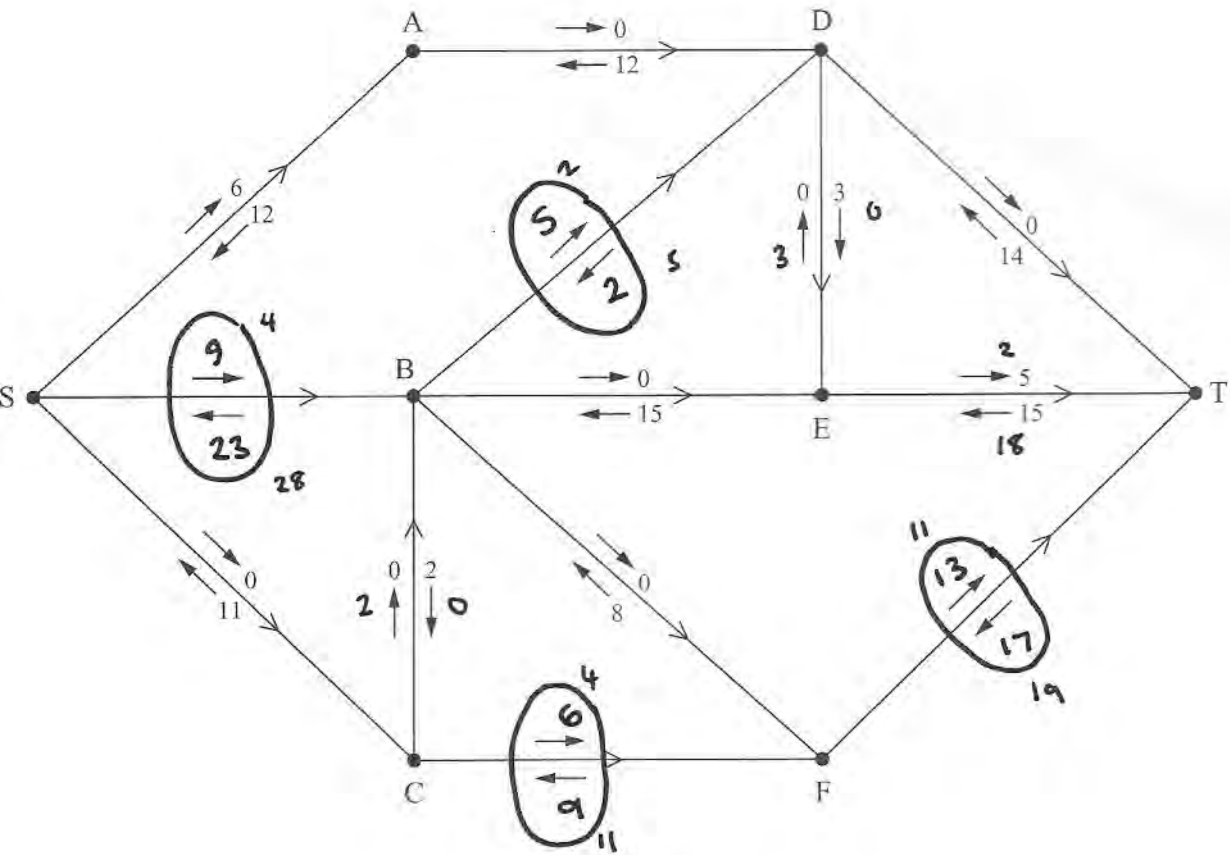

Diagram 1

SBDET (3) + 5 to the flow
SBCFT (2) flow now 51

(d)

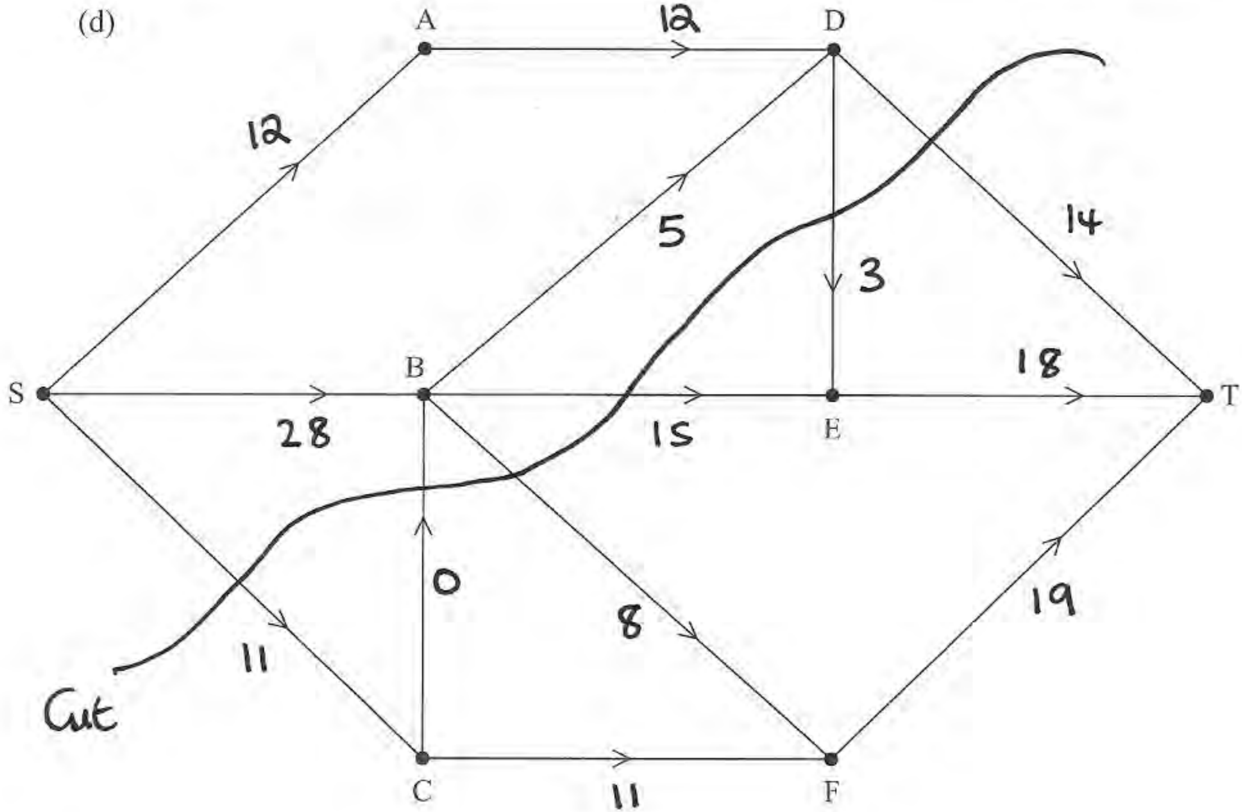

Diagram 2

(e)

Cut shown passes through saturated arcs:
SC; BF; BE; ED; DT and the empty arc back towards source BC ∴ min cut = 14 + 3 + 15 + 8 + 11 = 51
by mincut-maxflow theorem max flow = 51
∴ flow is maximal

7. Four workers, A, B, C and D, are to be assigned to four tasks, P, Q, R and S. Each worker is to be assigned to exactly one task and each task must be assigned to just one worker. The cost, in pounds, of using each worker for each task is given in the table below. The total cost is to be minimised.

	P	Q	R	S
A	23	41	34	44
B	21	45	33	42
C	26	43	31	40
D	20	47	35	46

Formulate the above situation as a linear programming problem. You must define your decision variables and make the objective function and constraints clear.

(Total 7 marks)

7. let $X_{ij} = \begin{cases} 1 & \text{if worker } i \text{ is allocated to task } j \\ 0 & \text{if worker } i \text{ is not allocated to task } j \end{cases}$

where worker $i \in \{A, B, C, D\}$ task $j \in \{P, Q, R, S\}$

Objective is to minimise cost, C (£) where

$$C = 23X_{AP} + 41X_{AQ} + 34X_{AR} + 44X_{AS}$$
$$+ 21X_{BP} + 45X_{BQ} + 33X_{BR} + 42X_{BS}$$
$$+ 26X_{CP} + 43X_{CQ} + 31X_{CR} + 40X_{CS}$$
$$+ 20X_{DP} + 47X_{DQ} + 35X_{DR} + 46X_{DS}$$

Subject to

$$\sum X_{Aj} = 1 \qquad \sum X_{iP} = 1$$
$$\sum X_{Bj} = 1 \qquad \sum X_{iQ} = 1$$
$$\sum X_{Cj} = 1 \qquad \sum X_{iR} = 1$$
$$\sum X_{Dj} = 1 \qquad \sum X_{iS} = 1$$

8. A company makes industrial robots.
They can make up to four robots in any one month, but if they make more than three they will have to hire additional labour at a cost of £400 per month.
They can store up to two robots at a cost of £150 per robot per month.
The overhead costs are £300 in any month in which work is done.

Robots are delivered to buyers at the end of each month. There are no robots in stock at the beginning of January and there should be none in stock after the April delivery.

The order book for robots is

Month	January	February	March	April
Number of robots required	2	2	3	4

Use dynamic programming to determine the production schedule which minimises the costs, showing your working in the table provided in the answer book.

(Total 12 marks)

TOTAL FOR PAPER: 75 MARKS

Stage	State	Action	Dest.	Value
1	2	2	0	300+300+ = 600 *
April	1	3	0	150+300+ = 450 *
	0	4	0	300+400 = 700 *
2	2	3	2	300+300+ +600 = 1200
March	2	2	1	300+300+ +450 = 1050 *
	2	1	0	300+300+ +700 = 1300
	1	4	2	150+300+400 +600 = 1450
	1	3	1	150+300+ +450 = 900 *
	1	2	0	150+300+ +700 = 1150
	0	4	1	300+400 +450 = 1150
	0	3	0	300+ +700 = 1000*
3	2	2	2	300+300+ +1050 = 1650
feb	2	1	1	300+300+ +900 = 1500
	2	0	0	300+ +1000 = 1300*
	1	3	2	150+300+ +1050 = 1500
	1	2	1	150+300+ +900 = 1350 *
	1	1	0	150+300+ +1000 = 1450
	0	4	2	300+400 +1050 = 1750
	0	3	1	300+ +900 = 1200*
	0	2	0	300+ +1000 = 1300
4	0	4	2	300+400+1300 = 2000
Jan	0	3	1	300+ +1350 = 1650
	0	2	0	300+ +1200 = 1500*

Month	Jan	Feb	March	April
Number of robots made	2	3	3	3

£1500