D2 June 2006 (Adapted)

1a) Bellman's - any part of an optimal path is itself optimal

b) minimax route - route is chosen so that the max arc is as small as possible

c) When planning stops for a long car journey you may want to choose the route so that the longest journey (time / miles) between stops is as small as possible.

2) let $X_{ij} = \begin{cases} 1 & \text{if worker } i \text{ is allocated to task } j \\ 0 & \text{if worker } i \text{ is \underline{not} allocated to task } j \end{cases}$

where $i \in \{P, Q, R\}$ workers
$j \in \{1, 2, 3\}$ tasks

objective is to minimise Cost C (£100s) where

$$C = 8X_{P1} + 7X_{P2} + 3X_{P3} + 9X_{Q1} + 5X_{Q2} + 6X_{Q3} + 10X_{R1} + 4X_{R2} + 4X_{R3}$$

Subject to

$$\sum X_{Pj} = 1 \qquad \sum X_{i1} = 1$$
$$\sum X_{Qj} = 1 \qquad \sum X_{i2} = 1$$
$$\sum X_{Rj} = 1 \qquad \sum X_{i3} = 1$$

3) Each activity is offered (visited) once during the week, before returning to the starting activity (node). We are attempting to complete this in the minimum time.

b) $B \xrightarrow{108} C \xrightarrow{54} D \xrightarrow{150} F \xrightarrow{68} T \xrightarrow{100} B$ 480min = 8hrs

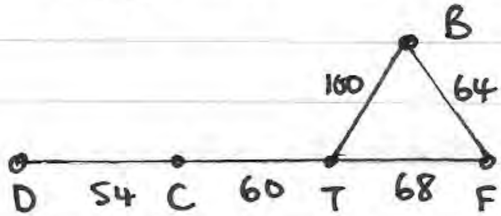
c) Nearest Neighbour

$B \xrightarrow{64} F \xrightarrow{68} T \xrightarrow{60} C \xrightarrow{54} D \xrightarrow{150} B$ 396min 6hrs 36m

d)

	C	D	F	T
~~C~~	~~-~~	~~54~~	~~104~~	~~60~~
~~D~~	54	~~-~~	~~150~~	~~102~~
F	104	150	-	68
T	60	102	68	-

Prim's
CD (54)
CT (60)
FT (68)
RMST = 182
+ BF (64)
+ BT (100)
= 346



lower bound = 346 (but not possible

346 < optimal time ≤ 396

4)

	H	P	R	W
A	3	5	11	9
B	3	7	8	25
C	2	5	10	7
D	8	3	7	6

0	2	8	6	-3
0	4	5	22	-3
0	3	8	5	-2
5	0	4	3	-3

reducing Rows

0	2	4	3
0	4	1	19
0	3	4	2
5	0	0	0
X	X	-4	-3

Reducing Columns

2 lines ∴ not optimal
smallest uncovered = 1

0	1	3	2
0	3	0	18
0	2	3	1
6	0	0	0

3 lines ∴ not optimal
smallest uncovered = 1

0	0	2	1
1	3	0	18
0	12	2	0
7	0	0	0

4 lines needed to cover zeros ∴ optimal

	H	P	R	W
A	0	0		
B			0	
C	0			0
D		0	0	0

A - H (3)
B - R (8)
C - W (7)
D - P (3)
21

A - P (5)
B - R (8)
C - H (2)
D - W (6)
21

∴ NOT UNIQUE

£21000

5)

Stage	State	Action	Dest.	Value
1	H	HT	T	4 *
	I	IT	T	3 *
	J	JT	T	12 *
	K	KT	T	20 *
2	D	DH	H	2+4 = 6
		DI	I	4 +3 = 7 *
	E	EH	H	3 +4 = 7 *
		EI	I	4 +3 = 7 *
	F	FJ	J	10 +12 = 22 *
		FK	K	-8 +20 = 12
	G	GJ	J	10 +12 = 22
	G	GK	K	17 +20 = 37 *
3	A	AD	D	3 +7 = 10
		AE	E	2 +7 = 9
		AF	F	-5 +22 = 17 *
	B	BD	D	3 + 7 = 10
		BE	E	2 +7 = 9
		BF	F	-6 +22 = 16 *
	C	CF	F	8 + 22 = 30 *
		CG	G	-15 + 37 = 22
4	S	SA	A	2 +17 = 19
		SB	B	3 +16 = 19
		SC	C	-10 +30 = 20 *

SCFJT (20) £20000

a) Degenerate when the number of occupied cells in the solution is not equal to columns + rows − 1

b) Dummy will be required if supply > demand

c)

	1	2	3 (Dummy)	Supply
A	62	47	0	15
B	61	48	0	12
C	68	58	0	17
Demand	16	11	17	

NW

	1	2	3	
A	15			15
B	1	11	0	12
C			17	17
	16	11	17	

	62	49	1
0	62	47	0
−1	61	48	0
−1	68	58	0

Shadow Costs

X	−2	−1
X	X	X
7	10	X

Improvement Indices

	1	2	3
A	15 −θ	+θ	
B	1 +θ	11 −θ	0
C			17

entering cell = A2

θ = 11

exiting cell = B2

4	11	
12		0
		17

new Solution

	62	47	1
0	62	47	0
−1	61	48	0
−1	68	58	0

Shadow Costs

X	X	−1
X	2	X
7	12	X

Improvement Indices

$4^{-\theta}$	11	$+\theta$
$12^{+\theta}$		$0-\theta$
		17

entering cell = A3
$\theta = 0$
exiting cell = B3

4	11	0
12		
		17

	62	47	0
0	62	47	0
-1	61	48	0
0	68	58	0

Shadow Costs

X	X	X
X	2	1
6	11	X

Improvement Indices

no negative Improvement index ⇒ Solution is optimal

Cost = £1497

7)

	B1	B2	B3
A1	5	7	2
A2	3	8	4
A3	6	4	9

V = value of game to A

A plays 1 prob = P_1
A plays 2 prob = P_2
A plays 3 prob = P_3

If B plays 1 $V = 5P_1 + 3P_2 + 6P_3$
B plays 2 $V = 7P_1 + 8P_2 + 4P_3$
B plays 3 $V = 2P_1 + 4P_2 + 9P_3$

∴ Objective, maximise $P = V$, $P - V = 0$ subject to

$$V - 5P_1 - 3P_2 - 6P_3 \leq 0$$
$$V - 7P_1 - 8P_2 - 4P_3 \leq 0$$
$$V - 2P_1 - 4P_2 - 9P_3 \leq 0$$
$$P_1 + P_2 + P_3 \leq 1$$
$$P_1, P_2, P_3 \geq 0$$

$$V - 5P_1 - 3P_2 - 6P_3 + r = 0$$
$$V - 7P_1 - 8P_2 - 4P_3 + s = 0$$
$$V - 2P_1 - 4P_2 - 9P_3 + t = 0$$
$$P_1 + P_2 + P_3 + u = 1$$
$$P - V = 0$$

b) no dominate row or column so game cannot be reduced, so remains a 3 variable problem.

c)

bv	V	p_1	p_2	p_3	r	s	t	u	Value	
r	$\boxed{1}$	-5	-3	-6	1	0	0	0	0	$\theta = 0 \div 1 = 0$
s	1	-7	-8	-4	0	1	0	0	0	$\theta = 0 \div 1 = 0$
t	1	-2	-4	-9	0	0	1	0	0	$\theta = 0 \div 1 = 0$
u	0	1	1	1	0	0	0	1	1	$\theta = 1 \div 0 = \infty$
P	-1	0	0	0	0	0	0	0	0	

d) Increase V, choose row 1

bv	V	p_1	p_2	p_3	r	s	t	u	Value	Row Opps	
V	1	-5	-3	-6	1	0	0	0	0	$R1 \div 1$	-ve
s	0	-2	-5	$\boxed{2}$	-1	1	0	0	0	$-R1$	$\theta = 0 \div 2$ *
t	0	3	-1	-3	-1	0	1	0	0	$-R1$	-ve
u	0	1	1	1	0	0	0	1	1	X	$\theta = 1 \div 1$
P	0	-5	-3	-6	1	0	0	0	0	$+R1$	

Increase p_3, choose row 2

	V	p_1	p_2	p_3	r	s	t	u	Value	Row Opps
V	1	-11	-18	0	-2	3	0	0	0	$+6\ \frac{1}{2}R2$
p_3	0	-1	$-5/2$	1	$-1/2$	$1/2$	0	0	0	$R2 \div 2$
t	0	0	$-17/2$	0	$-5/2$	$3/2$	1	0	0	$+3\ \frac{1}{2}R2$
u	0	2	$7/2$	0	$1/2$	$-1/2$	0	1	1	$-\ \frac{1}{2}R2$
P	0	-11	-18	0	-2	3	0	0	0	$+6\ \frac{1}{2}R2$

8) $7x + 10y + 10z + r = 3600$
$6x + 9y + 12z + s = 3600$
$2x + 3y + 4z + t = 2400$
$P - 35x - 55y - 60z = 0$

b.v.	x	y	z	r	s	t	Value	Row Operations
r	7	10	10	1	0	0	3600	$\theta = 3600 \div 10 = 360$
s	6	9	$\boxed{12}$	0	1	0	3600	$\theta = 3600 \div 12 = 300$ *
t	2	3	4	0	0	1	2400	$\theta = 2400 \div 4 = 600$
P	-35	-55	-60	0	0	0	0	

Increasing z

b.v.	x	y	z	r	s	t	Value	Row Operations
r	7	10	10	1	0	0	3600	R1
	$1/2$	$3/4$	1	0	$1/12$	0	300	R2 ÷ 12
t	2	3	4	0	0	1	2400	R3
P	-35	-55	-60	0	0	0	0	R4

b.v.	x	y	z	r	s	t	Value	Row Operations	
r	2	$\boxed{5/2}$	0	1	$-5/6$	0	600	R1 − 10R2	$\theta = 600 \div \frac{5}{2} = 240$ *
z	$1/2$	$3/4$	1	0	$1/12$	0	300	R2	$\theta = 300 \div \frac{3}{4} = 400$
t	0	0	0	0	$-1/3$	1	1200	R3 − 4R2	$\theta = \infty$
P	-5	-10	0	0	5	0	18000	R4 + 60R2	

Increasing y

b.v.	x	y	z	r	s	t	Value	Row Operations
	$4/5$	1	0	$2/5$	$-1/3$	0	240	R1 × $\frac{2}{5}$
z	$1/2$	$3/4$	1	0	$1/12$	0	300	R2
t	0	0	0	0	$-1/3$	1	1200	R3
P	-5	-10	0	0	5	0	18000	R4

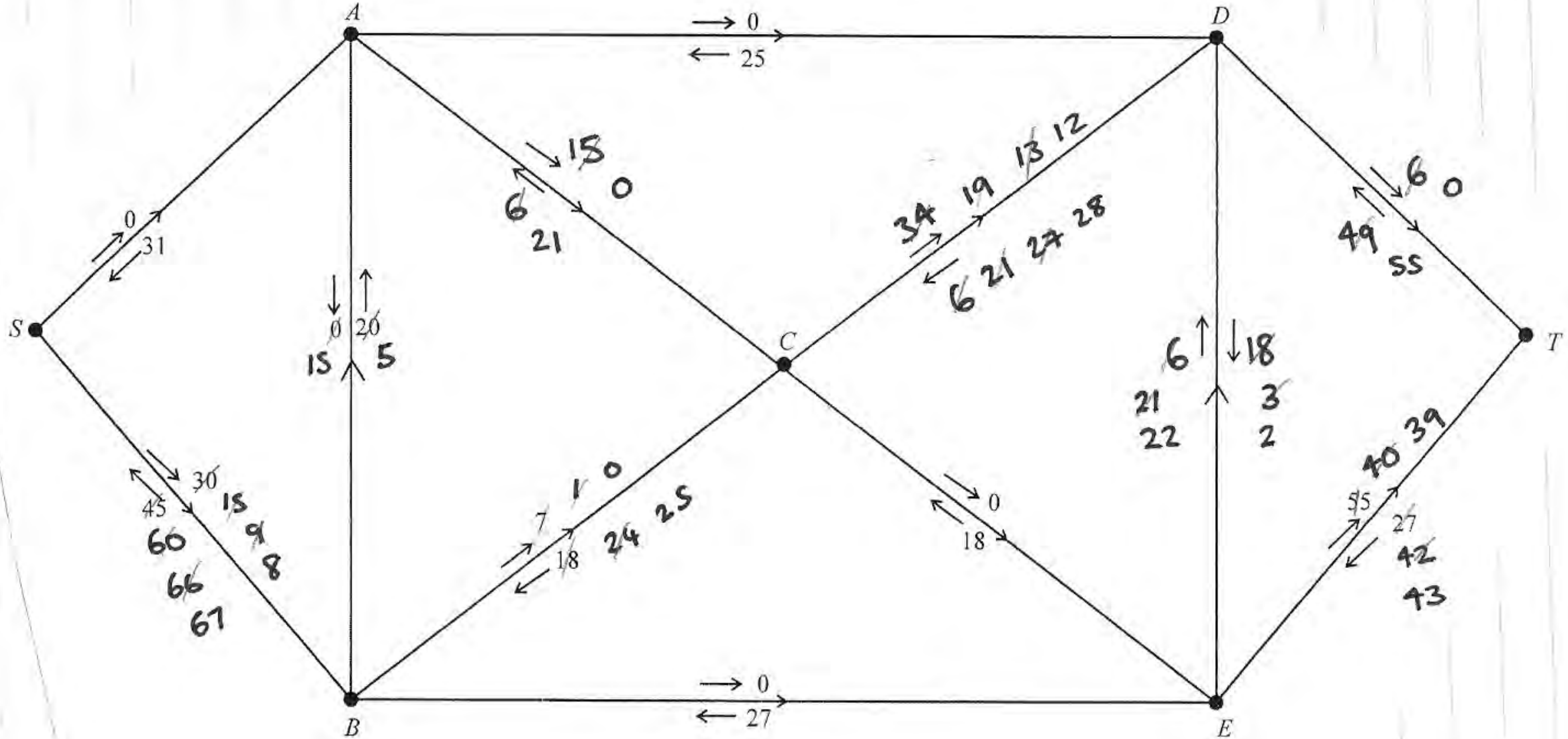
b.v.	x	y	z	r	s	t	Value	Row Operations
y	$4/5$	1	0	$2/5$	$-1/3$	0	240	R1
z	$-1/10$	0	1	$-3/10$	$1/3$	0	120	$R2 - \frac{3}{4}R1$
t	0	0	0	0	$-1/3$	1	1200	R3
P	3	0	0	4	$5/3$	0	20400	$R4 + 10R1$

$x = 0 \quad y = 240 \quad z = 120 \quad r = 0 \quad s = 0 \quad t = 1200$
$P = 20400$

9) Inital flow $= 31 + 45 = 76$

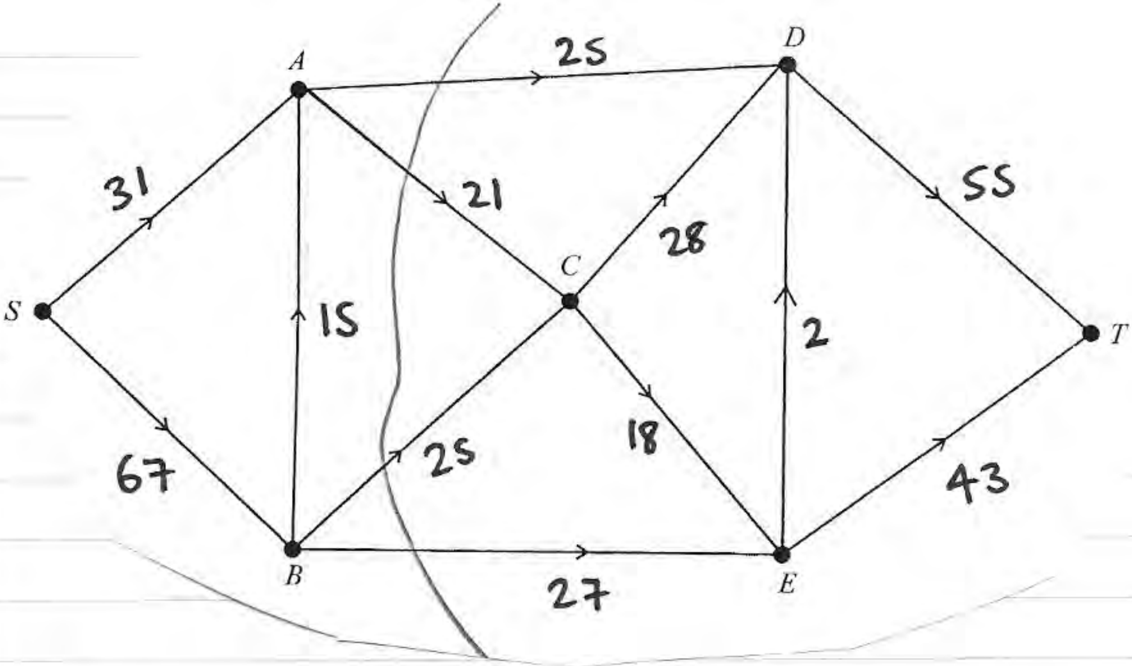
$C_1 = 31 + 20 + 25 + 27 = 103$

$C_2 = 25 + 21 + 25 + 24 + 82 = 177$



SBACDET (15)
SBCDT (6)
SBCDET (1)

22



Cut through AD, AC, BC, BE only passes through saturated arcs $\therefore$ min cut = 25+21+25+27
min cut = 98

$\therefore$ by min cut-max flow theorem max flow = 98

$\therefore$ flow is maximal