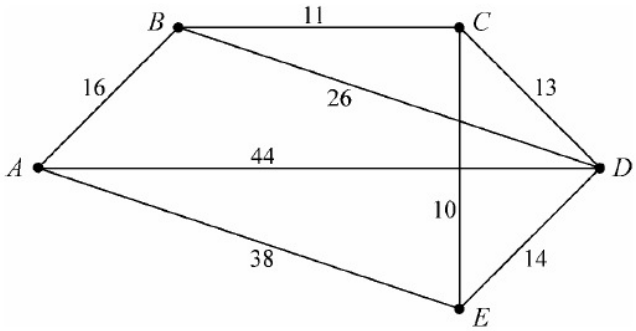
Decision Mathematics 1
Dijkstra's, Chinese Postman and Nearest Neighbour Algorithm

Notes:

- Dijkstra's Algorithm finds the shortest distance between a given node and all other nodes in the network.
- The Chinese Postman (Route Inspection) Algorithm finds a closed trail of minimum weight e.g. a route that starts and finished as the same node and visits every arc. This algorithm uses the theory of Eulerian Graphs, that every node needs to become of even order.
- The Nearest Neighbour (Travelling Salesman) Algorithm finds a tour of minim weight e.g. the shortest distance to visit every node and end up back where you started.

Specimen Paper – Question 6:

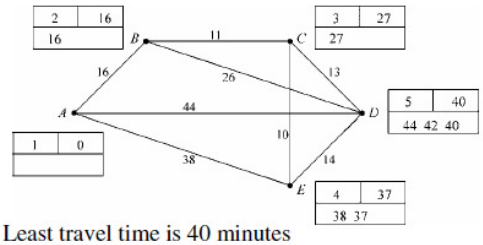
The diagram shows a simplified version of an orienteering course. The vertices represent checkpoints and the weights on the arcs show the travel times between checkpoints, in minutes.



(i) Use Dijkstra's algorithm, **starting from checkpoint *A***, to find the least travel time from *A* to *D*. You must show your working, including temporary labels, permanent labels and the order in which permanent labels were assigned. Give the route that takes the least time from *A* to *D*. [6]

(ii) By using an appropriate algorithm, find the least time needed to travel **every arc** in the diagram starting and ending at *A*. You should show your method clearly. [6]

(iii) Starting from *A*, apply the nearest neighbour algorithm to the diagram to find a cycle that visits every checkpoint. Use your solution to find a path that visits every checkpoint, starting from *A* and finishing at *D*. [3]

6	(i)			
			M1	For correct use of temporary labels
			M1	For updating E and D
			A1	For all permanent labels correct
			B1	For correct order of assignment stated
		Least travel time is 40 minutes	B1	For correct value 40
		Route is A–B–C–D	B1 **6**	For correct route
	(ii)	The Route Inspection algorithm is used	B1	For stating or implying the correct algorithm
		A, B, C and E are odd nodes	B1	For identifying the odd nodes
		$AB = 16$ $AC = 27$ $AE = 37$ $CE = \underline{10}$ $BE = \underline{21}$ $BC = \underline{11}$ $\underline{26}$ $\underline{48}$ $\underline{48}$	M1	For pairing odd nodes correctly
		Double up on AB and CE	M1	For selecting appropriate pair for doubling
		Sum of arcs is 172	M1	For adding weights on all the arcs
		Hence shortest time is $172 + 26 = 198$ minutes	A1 **6**	For correct value 198
	(iii)	Nearest neighbour algorithm gives A–B–C–E–D–A	M1	For starting the algorithm correctly, up to C
			A1	For the correct cycle A–B–C–E–D–A
		Hence required path is A–B–C–E–D	B1 **3**	For a correct path
			15	

June 2007, Question 6

The table shows the distances, in miles, along the direct roads between six villages, A to F. A dash (–) indicates that there is no direct road linking the villages.

	A	B	C	D	E	F
A	–	6	3	–	–	–
B	6	–	5	6	–	14
C	3	5	–	8	4	10
D	–	6	8	–	3	8
E	–	–	4	3	–	–
F	–	14	10	8	–	–

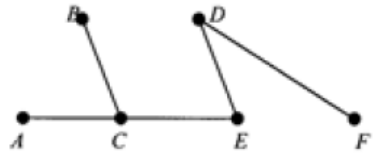
(i) On the table in the insert, use Prim's algorithm to find a minimum spanning tree. Start by crossing out row A. Show which entries in the table are chosen and indicate the order in which the rows are deleted. Draw your minimum spanning tree and state its total weight. [6]

(ii) By deleting vertex B and the arcs joined to vertex B, calculate a lower bound for the length of the shortest cycle through all the vertices. [3]

(iii) Apply the nearest neighbour method to the table above, starting from F, to find a cycle that passes through every vertex and use this to write down an upper bound for the length of the shortest cycle through all the vertices. [4]

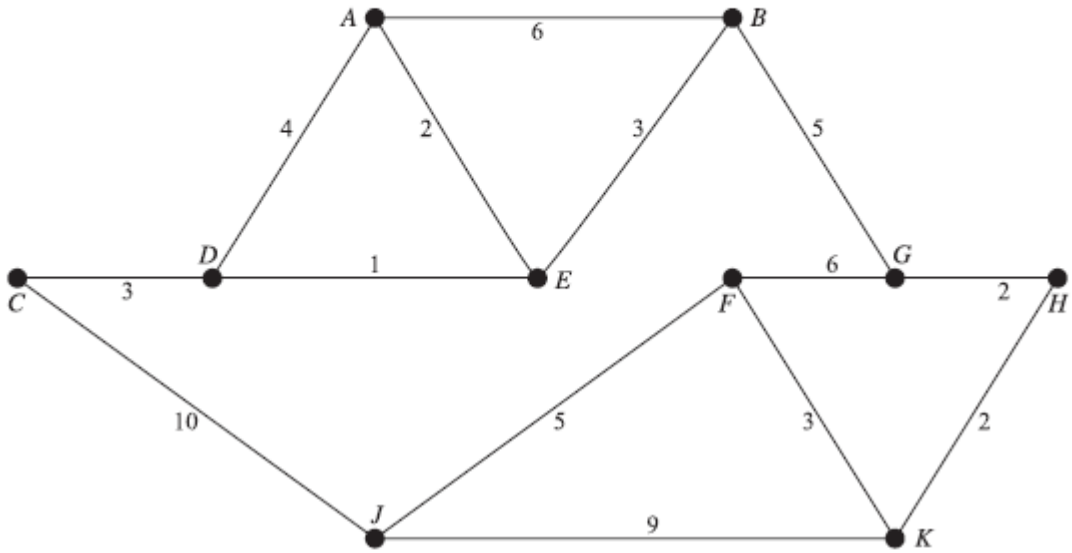
6	(i)			ANSWERED ON INSERT

	1	5	2	4	3	6
	A	*B*	*C*	*D*	*E*	*F*
A	-	6	3	-	-	-
B	6	-	**5**	6	-	14
C	**3**	5	-	8	4	10
D	-	6	8	-	**3**	8
E	-	-	**4**	3	-	-
F	-	14	10	**8**	-	-

	M1	For choosing row *C* in column *A*
	M1 dep	For choosing more than one entry from column *C*
	A1	For correct entries chosen
Order: *A C E D B F*	B1	For correct order, listed or marked on arrows or table, or arcs listed *AC CE ED CB DF*
Minimum spanning tree:		
	B1	For tree (correct or follow through from table, provided solution forms a spanning tree)
Total weight: 23 miles	B1	For 23 (or follow through from table or diagram, provided solution forms a spanning tree)
	6	
(ii) MST for reduced network = 18	M1	For their 18 seen or implied
Two shortest arcs from *B* = 5 + 6 = 11	M1	For 11 seen or implied
Lower bound = 29 miles	A1 **3**	For 29 (cao)
(iii) *F – D – E – C – A – B – F*	M1	For *F–D–E–C–A–B*
	A1	For correct tour
8 + 3 + 4 + 3 + 6 + 14	M1 **4**	For a substantially correct attempt at sum
= 38 miles	A1 **13**	For 38 (cao)

June 2008, Question 4

The vertices in the network below represent the rooms in a house. The arcs represent routes between rooms, and the weights on the arcs represent distances in metres.

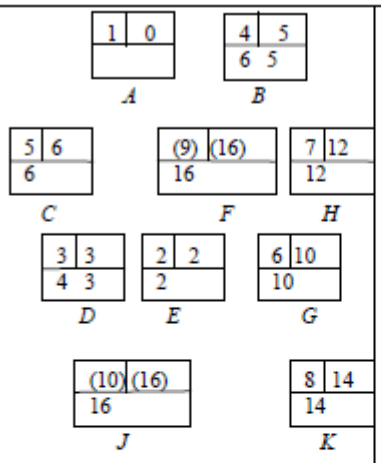


(i) On the diagram in the insert, use Dijkstra's algorithm to find the shortest path from A to K. You must show your working, including temporary labels, permanent labels and the order in which permanent labels are assigned. Write down the route of the shortest path from A to K and give its length in metres. [7]

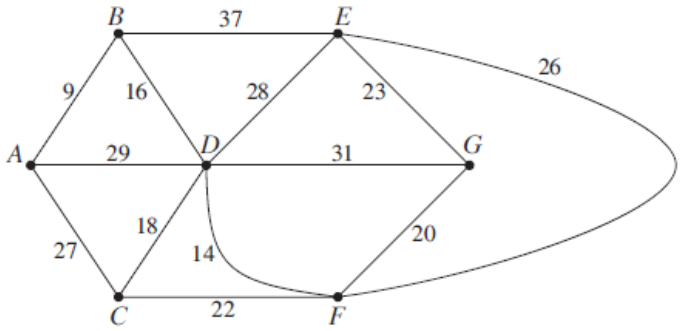
A locked door blocks the route CJ, so this arc cannot be used.

(ii) Use your answer to part **(i)** to find the route of the shortest path from A to J and its length in metres. [2]

(iii) Alterations mean that the length of route FJ changes from its current value of 5 metres. By how much would it have to change if the route of the shortest path from A to J, not using CJ, changes from that found in part **(ii)**? [2]

4	(i)	 Route = $A - E - B - G - H - K$ Length = 14 metres	M1 M1 A1 B1 B1 B1 B1	Both 6 and 5 shown at B All temporary labels correct including F and J No extra temporary labels All permanent labels correct (may omit F and/or J) cao Order of labelling correct (may omit F and/or J, may reverse F and J) cao $A - E - B - G - H - K$ cao 14 cao	[7]
	(ii)	Without using CJ: Route = $A - E - B - G - F - J$ Length = 21 metres	 B1 B1	Follow through their (i) $A - E - B - G - F - J$ 21	[2]
	(iii)	More than 2 metres (Answer of 'more than 7 metres' or '7 metres' ⇒ M1, A0)	M1 A1	2 (cao) More than, or equivalent (Answer of 3 or ≥ 3 ⇒ SC1)	[2]
				Total =	**11**

January 2009, Question 3



(i) This diagram shows a network. The insert has a copy of this network together with a list of the arcs, sorted into increasing order of weight. Use Kruskal's algorithm on the insert to find a minimum spanning tree for this network. Draw your tree and give its total weight. **[5]**

(ii) Use your answer to part **(i)** to find the weight of a minimum spanning tree for the network with vertex E, and all the arcs joined to E, removed. Hence find a lower bound for the travelling salesperson problem on the original network. **[3]**

(iii) Show that the nearest neighbour method, starting from vertex A, fails on the original network. **[2]**

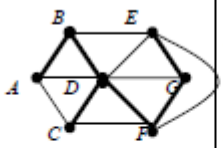
(iv) Apply the nearest neighbour method, starting from vertex B, to find an upper bound for the travelling salesperson problem on the original network. **[3]**

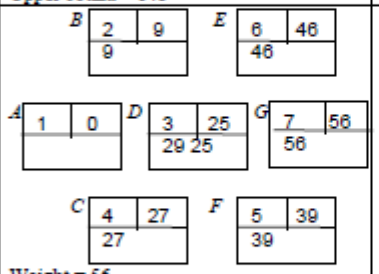
(v) Apply Dijkstra's algorithm to the copy of the network in the insert to find the least weight path from A to G. State the weight of the path and give its route. **[6]**

(vi) The sum of the weights of all the arcs is 300.

Apply the route inspection algorithm, showing all your working, to find the weight of the least weight closed route that uses every arc at least once. The weights of least weight paths from vertex A should be found using your answer to part **(v)**; the weights of other such paths should be determined by inspection. **[4]**

ANSWERED ON INSERT

3	(i)	$AB = 9$ $DF = 14$ $BD = 16$ $CD = 18$ $FG = 20$ ~~$CF = 22$~~ $EG = 23$ ~~$EF = 26$~~ ~~$AC = 27$~~ ~~$DE = 28$~~ ~~$AD = 29$~~ ~~$DG = 31$~~ ~~$BE = 37$~~	 Total weight = 100	M1 A1 M1 A1 B1	Not selecting CF (working seen on list) Selecting correct arcs (working seen on list) A spanning tree drawn Correct (minimum) spanning tree drawn 100 cao	[5]

	(ii)	Delete EG from spanning tree $100 - 23 = 77$ Two shortest arcs from E are EG and EF $77 + 23 + 26 = 126$ Lower bound = 126	 B1 M1 A1	Follow through from part (i) if possible Weight of MST on reduced network Adding two shortest arcs to MST 126 cao	 **[3]**
	(iii)	$A - B - D - F - G - E$ – stall Misses out vertex C	M1 A1	$A - B - D - F - G - E$ <u>Cannot continue</u> because B, D and F have already been visited	 **[2]**
	(iv)	$B - A - C - D - F - G - E - B$ Upper bound = 148	M1 A1 B1	Tour starts $B - A - C - D - F -$ Correct tour, starting and ending at B 148 cao	 **[3]**
	(v)	 Weight = 56 Route = A - B - D - G	 M1 A1 B1 B1 B1	(Accept correct working starting from G, if seen) At least three sets of temporary labels correct, with no extras Temporary labels all correct, with no extras Permanent labels correct Order of labelling (correct or follow through their permanent labels) 56 cao A - B - D - G cao	 **[4]** **[2]**
	(vi)	A, B, C and G are odd $AB = 9$ $AC = 27$ $AG = 56$ $CG = \underline{42}$ $BG = \underline{47}$ $BC = \underline{34}$ 51 74 90 Repeat AB and CG (C - F - G) = 51 Weight = 300 + 51 = 351	B1 M1 A1 B1	Identifying or using A, B, C, G (seen) At least one correct pairing seen or total seen (not just six weights) All three totals correct, or explanation of how it is known that other pairings are too long 351 cao	 **[4]**
				Total =	**23**