Review Exercise 2

1 a i Shortest path through A is $18 + y$ or 26, both of which are greater than 17.
Shortest path through C is 23, which is greater than 17. So shortest path cannot go through A or C.

ii Shortest path must go through B
$SBDT = 13 + x$
$13 + x = 17$
$x = 4$

b If $y = 0$ shortest path is $SADT = 18$
If $y = 5$ shortest path is $SCDT = 23$
so range is 18 to 23.

c For example, a person seeking the quickest route from home to work through a city. The arcs are the roads that may be chosen, the number the time, in minutes, to journey along that road. The nodes represent junctions.

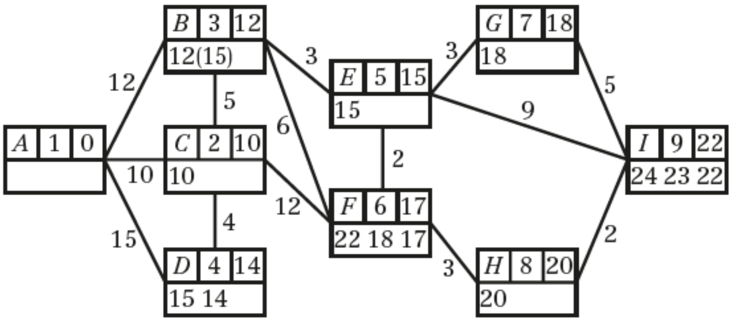
2 a Odd vertices are B_1, B_2, E, G
$B_1B_2 + EG = 65 + 18 = 83$
$B_1E + B_2G = 41 + 42 = 83$
$B_1G + B_2E = 26 + 30 = 56$
Repeat B_1D, DG, B_2A, AE
Route: For example,
$FAEAB_2ACEFGDHGDB_1DF$
(All correct routes have 17 letters in their 'word')
Length = 129 + 56 = 185 km

b Now only the route between E and G needs repeating
so repeat $EF + FG = 18$
length of new route = 129 + 18
= 147 km

3 a All arcs are to be traversed twice, this is, in effect, repeating each arc. So all valencies are even.

b e.g. $ABDGFGDCEAECAFEFBFABDCA$
(all correct routes will have 23 letters in their name)
length = 2×6 = 12 km

4 **a**



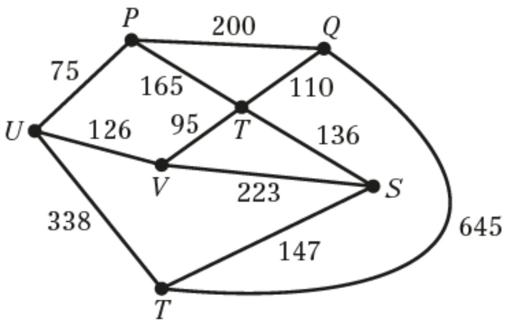
Shortest route is *ABEFHI* length 22 km

b **i** Odd vertices are *A* and *I* (only), so we need to repeat the shortest route from *A* to *I*. This was found in **a**.
So repeat *AB*, *BE*, *EF*, *FH*, *HI*.

ii For example *ABCADCEHIHEFIGFEBFBA*
(20 letters in route)

iii 91 + 22 = 113 km

5



a Total length = 2260 m
Odd nodes *P, Q, S, T, U, V*
T and *P* remain odd.
QS + *UV* = 246 + 126 = 372 ← least weight
QU + *SV* = 275 + 223 = 498
QV + *SU* = 205 + 349 = 554
QS and *UV* gives the shortest pairing.
Roads to be traversed twice: *QR*, *RS*, *UV*

b Total of roads traversed twice 110 + 136 + 126 = 372
Shortest route is 2260 + 372 = 2632 m

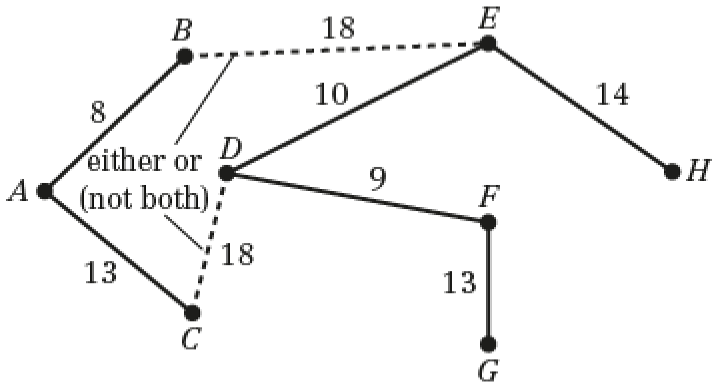
6 **a** Odd valencies are at *A, B, C, D, F, G*
Route starts at *A* and finishes at *G* so these can remain odd.
Choose pairings of remaining odd vertices *B, C, D, F*
By inspection, these path lengths are:
BC + *DF* = 0.8 + 1.7 = 2.5
BD + *CF* = 1.3 + 2.3 = 3.6
BF + *CD* = 1.5 + 0.7 = 2.2 ← least weight
Repeating BF and *CD* minimises the total distance travelled.
Length = 9.5 + 2.2 = 11.7 km

b *ABCAGBDCDEFBFG* (14 letters in route)

6 c Repeating AC and $BF = 2.1$
Minimum distance = 11.6 km
The engineer is correct. His new route is 0.1 km shorter.

7 a In the *practical* T.S.P each vertex must be visited *at least once*
In the *classical* T.S.P. each vertex must be visited *exactly once*

b $AB,\ DF,\ DE,\ (\text{reject } EF),\left\{\begin{matrix} FG \\ AC \end{matrix}\right\},\ EH,\left\{\begin{matrix} DC \\ \text{or} \\ BE \end{matrix}\right\}$



c Initial upper bound = 2×85 = 170 km

d When CD is part of tree
Use GH (saving 26) and BD (saving 19) giving a new upper bound of 125 km
Tour $ABDEHGFDCA$
e.g. when BE is part of tree
Use CG (saving 40) giving a new upper bound of 130 km
Tour $ABEHEDFGCA$

8 a

	A	*B*	*C*	*D*	*E*	*F*
A	–	20	30	32	12	15
B	20	–	10	25	32	16
C	30	10	–	15	35	19
D	32	25	15	–	20	34
E	12	32	35	20	–	16
F	15	16	19	34	16	–

Each row shows the shortest route.
The first row shows the shortest route starting at A. There are direct routes from AB, AE and AF and these are the shortest routes. AC (30) is by observation using ABC and AD (32) is by observation using AED.

b AE (12), EF (16), FB (16), BC (10), CD (15), DA (32)
101 km tour $AEFBCDA$

c In the original network AD is not a direct path. The tour becomes $AEFBCDEA$.

8 d e.g.

$$\left.\begin{array}{l} B\ C\ D\ E\ A\ F\ B \\ C\ B\ F\ A\ E\ D\ C \\ D\ C\ B\ F\ A\ E\ D \\ E\ A\ F\ B\ C\ D\ E \\ F\ A\ E\ D\ C\ B\ F \end{array}\right\}\ \text{length } 88$$

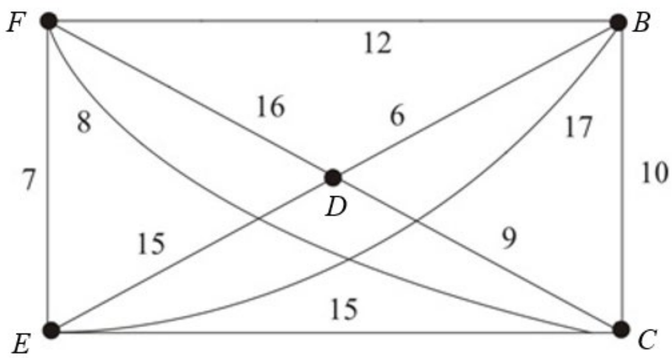
9 a i Minimum connector using Prim: AC, CB, CD, CE
length = 98 + 74 + 82 + 103 = 357 {1, 3, 2, 4, 5}
So upper bound = 2 × 357 = 714

ii $A\underline{(98)}\ C\underline{(74)}\ A\underline{(131)}\ D\underline{(134)}\ E\underline{(115)}A$
length = 98 + 74 + 131 + 134 + 115 = 552

b Residual minimum connector is AC, CB, CD length 254
Lower bound = 254 + 103 + 115 = 472

c 472 ⩽ solution ⩽ 552

10 a Deleting vertex A we obtain

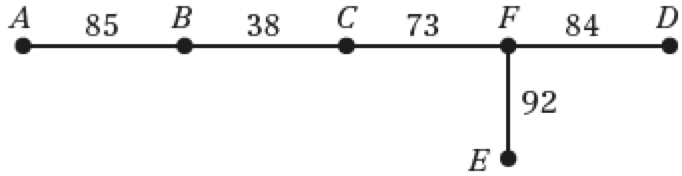


By Kruskal's algorithm an MST is $DB(6)$, $EF(7)$, $CF(8)$, $DC(9)$ of weight 30
The two edges of least weight at A are $AE(7)$ and $AD(8)$
∴ A lower bound is 30 + 8 + 7 = 45

b i A – nearest neighbor $E(7)$
E – nearest neighbor $F(7)$
F – nearest neighbor $C(8)$
C – nearest neighbor $D(9)$
D – nearest neighbor $B(6)$
Complete tour with $BA(12)$
$AEFCDBA$ length 49

ii Choose a tour that does not use AB
e.g. $DB(6)$ $BC(10)$, $CF(8)$, $FE(4)$, $EA(4)$
Complete with $AD(8)$, $DBCFEAD$.
Total weight 46

11 a Order of arcs: AB, BC, CF, FD, FE

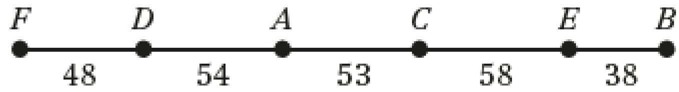


b i $2 \times 372 = 744$

ii e.g. AD saves 105 giving 639.
or AE saves 180 giving 564.
AF saves 96 giving 648.
DE saves 66 giving 678.

c Residual M.S.T.
AB, BC, AE, ED
Lower bound = 341 + 73 + 84
= 498

12 a $AC(53)$, $AD(54)$, $DF(48)$, $CE(58)$, $EB(38)$



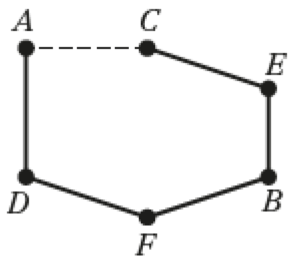
b i M.S.T. $XZ = 251 \times 2 = 502$

ii Finding a shortcut to below 360, e.g. FB =100 shortens by 151 so we get 251 + 100 = 351.

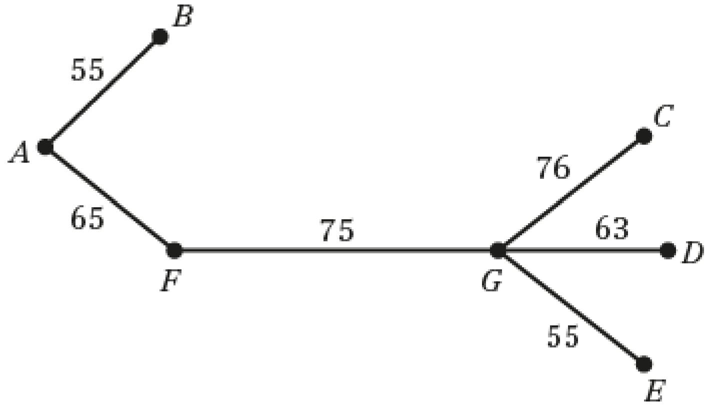
c M.S.T. is DF, CE, EB, FB length 244
The 2 shortest arcs are AC (53) and AD (54) giving a total of 351

d The optimal solution is 351 and is $ACEBFDA$.

e



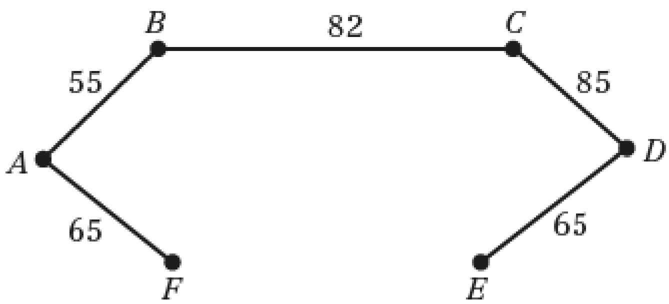
13 a Label column A, delete row A.
Scan all labelled columns and choose the least number.
Add that new vertex to the tree.
Label the new vertex's column and delete its row
Repeat the 3 steps until all vertices added.
Applying algorithm order of vertex selection A, B, F, G, E, D, C.



b Initial upper band = 2 × 389 = 778 km

c Reducing upper bound by short cuts
e.g. Using BC = 82 instead of $BA + AF + FG + GC$ leaves an upper bound of 589
Lists new route e.g. $ABCGDGEGFA$
States revisited vertices e.g. G

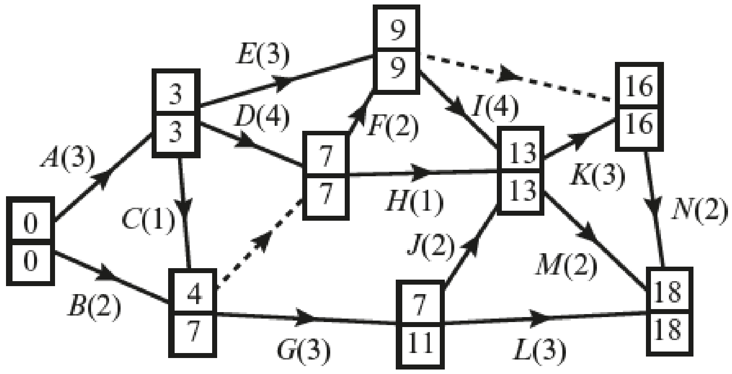
d



e Lower bound = 352 + GD + GE
= 352 + 63 + 55
= 470 km

f e.g. Use GE and GF (rather than GD)
length = 352 + 55 + 75 = 482 km
Route $ABCDEGFA$

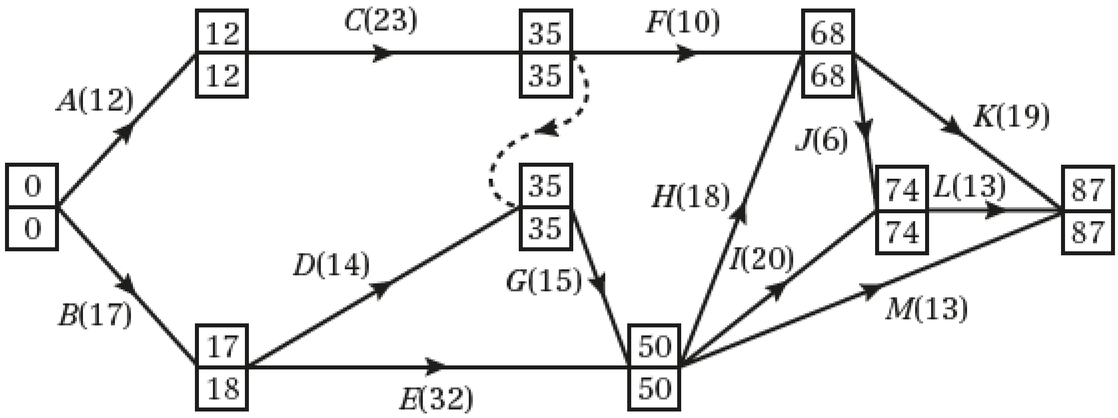
14 a



b 18 days

c *ADFIKN*

15 a



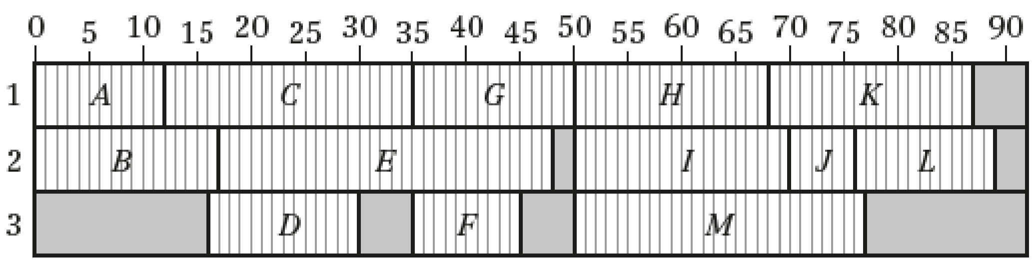
b *A*, *C*, *G*, *H*, *J*, *K* and *L*

All critical activities have a zero total float.

c Total float = 35 – 17 – 14 = 4

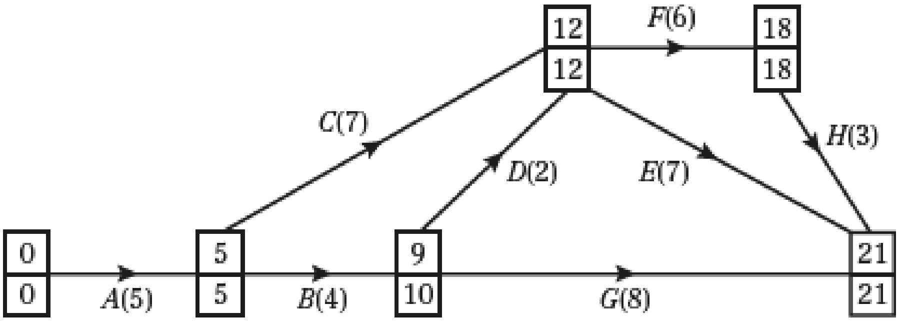
d *Either* $\frac{226}{87} = 2.6$ (1 d.p.) so at least 3 workers needed (here 226 is the total number of hours required for all the activities) *or* 69 hours into the project activities *J*, *K*, *I* and *M* *must* be happening so at least 4 workers will be needed.

e



New shortest time is 89 hours.

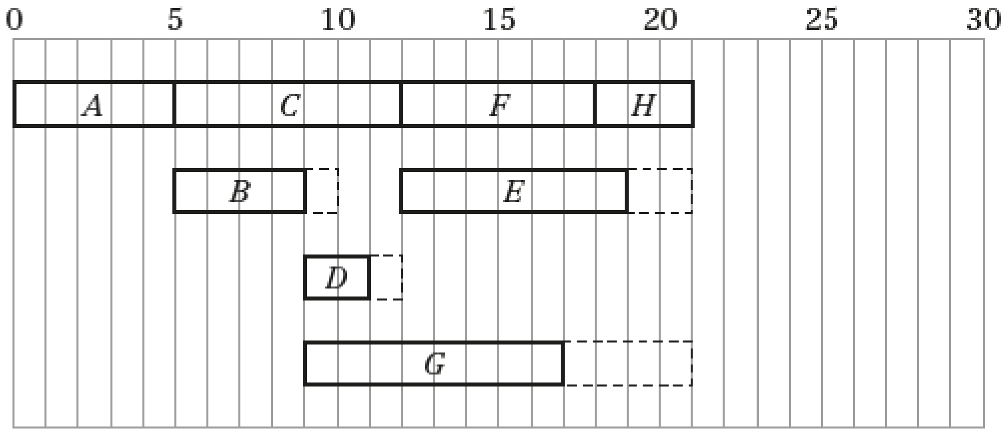
16 a



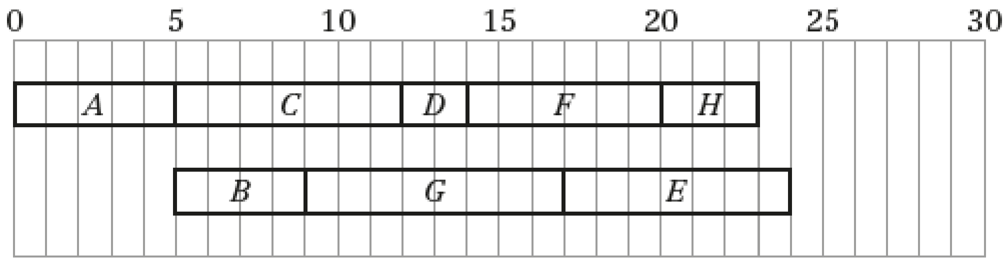
b Critical activities: A, C, F and H; length of critical path = 21

c Total float on $B = 10 - 5 - 4 = 1$ Total float on $E = 21 - 12 - 7 = 2$
Total float on $D = 12 - 9 - 2 = 1$ Total float on $G = 21 - 9 - 8 = 4$

d



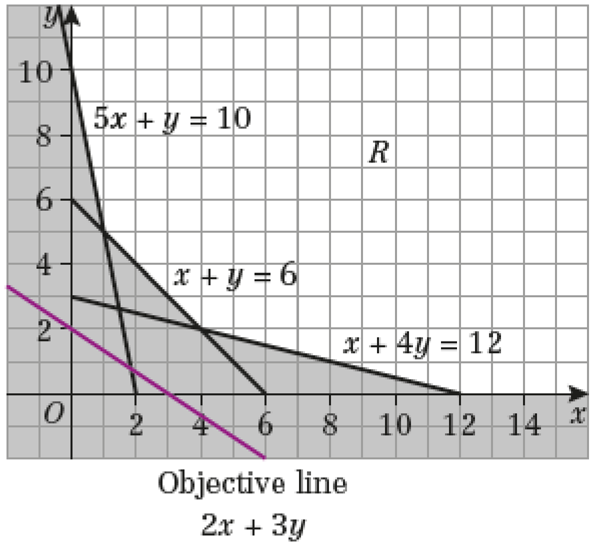
e For example;



Minimum time for 2 workers is 24 days.

17 a Chemical A $5x + y \geqslant 10$
Chemical B $2x + 2y \geqslant 12$ $[x + y \geqslant 6]$
Chemical C $\frac{1}{2}x + 2y \geqslant 6$ $[x + 4y \geqslant 12]$
$x, y \geqslant 0$

17 b



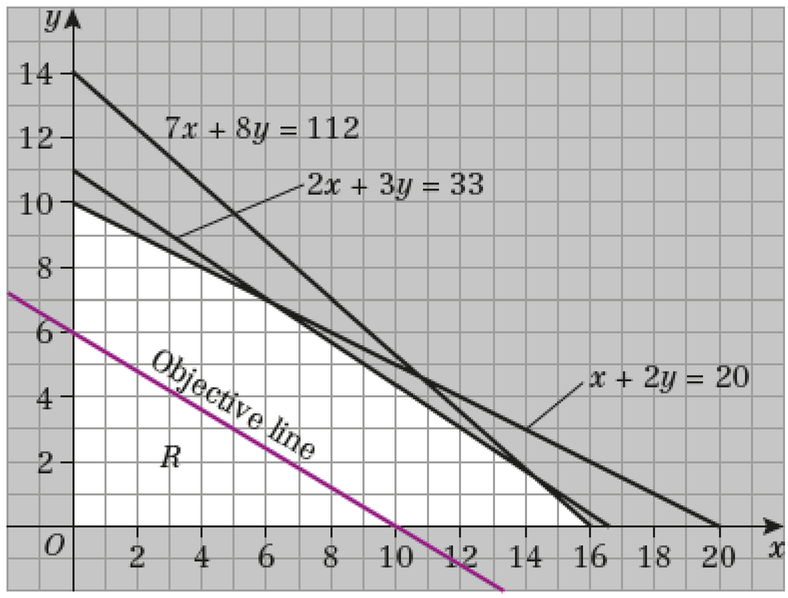
c $T = 2x + 3y$

d $(x, y) = (4,2)$ $T = 14$

18 a Maximise $P = 300x + 500y$

b Finishing $3.5x+4y \leqslant 56 \Rightarrow 7x+8y \leqslant 112$ (o.e.)
Packing $2x+4y \leqslant 40 \Rightarrow x+2y \leqslant 20$ (o.e.)

c



d For example, *point testing*

- Test all corner points in feasible region.
- Find profit at each and select point yielding maximum.

profit line

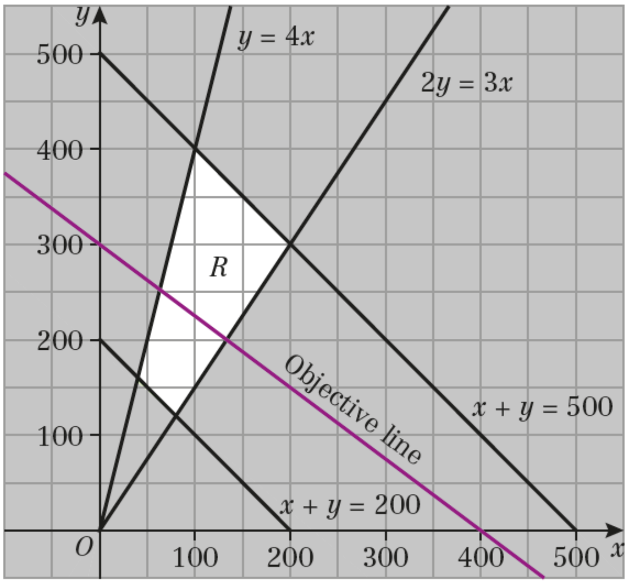
- Draw profit lines.
- Select point on profit line furthest from the origin.

18 e Using a correct, complete method.
Making 6 Oxford and 7 York gives a profit = £5300
$(6, 7) \rightarrow 5300$ $(14.4, 1.4) \xrightarrow{\text{integer}} (14, 1) \rightarrow 4700$ $(16, 0) \rightarrow 4800$
$(0, 10) \rightarrow 5000$

f The line $3.5x + 4y = 49$ passes through (6,7) so reduce *finishing* by 7 hours.

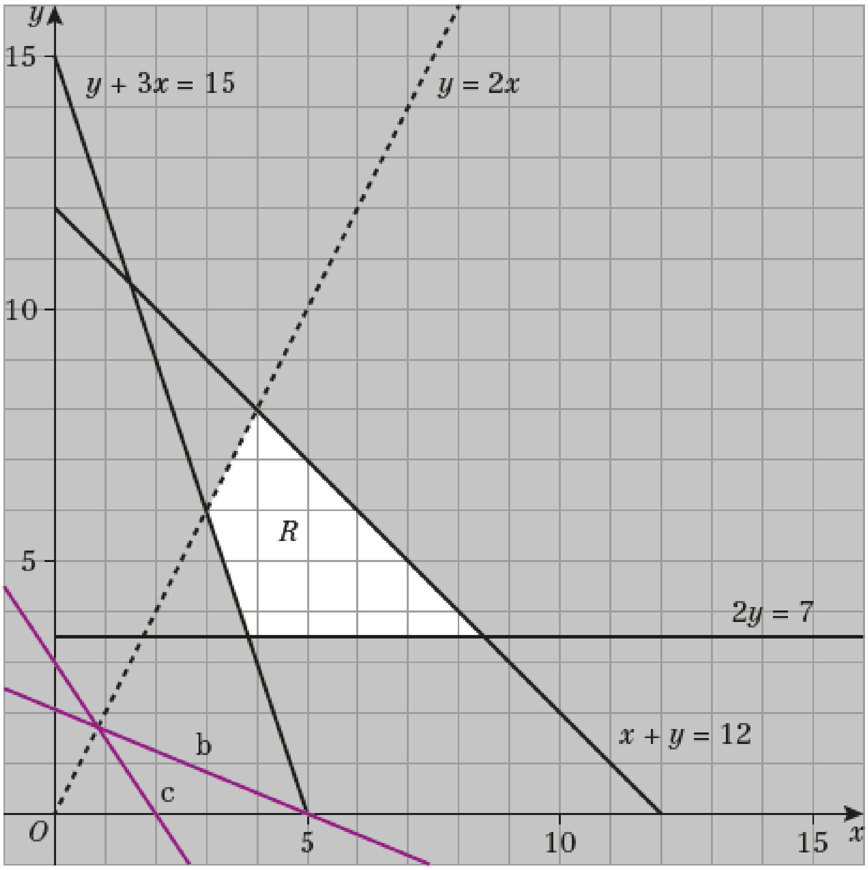
19 a Let badge 1 be x and badge 2 be y
$P = 30x + 40y$
$x + y \geq 200$
$x + y \leq 500$
$x \geq 0.2(x + y) \Rightarrow 4x \geq y$
$x \leq 0.4(x + y) \Rightarrow 3x \leq 2y$
$x \geq 0,\ y \geq 0$

b



c Testing the vertices:
(40, 160) = 7600
(80, 120) = 7200
(100, 400) = 19 000
(200, 300) = 18 000
Maximum profit of £190 at (100, 400) so they should make 100 of badge 1 and 400 of badge 2.

20 a



b Visible use of objective line method – objective line drawn or vertex testing.

$\left[\left(3\frac{5}{6}, 3\frac{1}{2}\right) \to 25\frac{1}{6}\left(8\frac{1}{2}, 3\frac{1}{2}\right) \to 34\frac{1}{2}(4, 8) \to 48(3, 6) \to 36\right]$

Optimal point $\left(3\frac{5}{6}, 3\frac{1}{2}\right)$ with value $25\frac{1}{6}$

c Visible use of objective line method – objective line drawn, or vertex testing – all 4 vertices tested.

$\left(3\frac{5}{6}, 3\frac{1}{2}\right)$ not an integer try (4,4) → 20 (4,8) → 28

$\left(8\frac{1}{2}, 3\frac{1}{2}\right)$ not an integer try (8,4) → 32 (3,6) → 21

Optimal point (8,4) with value £32, so Becky should use 4 kg of bird feeder and 3.5 kg of bird table food.

21 a Objective: maximise $P = 0.4x + 0.2y$ $(P = 40x + 20y)$
subject to:

$$x \leqslant 6.5$$
$$y \leqslant 8$$
$$x + y \leqslant 12$$
$$y \leqslant 4x$$
$$x,\ y \geqslant 0$$

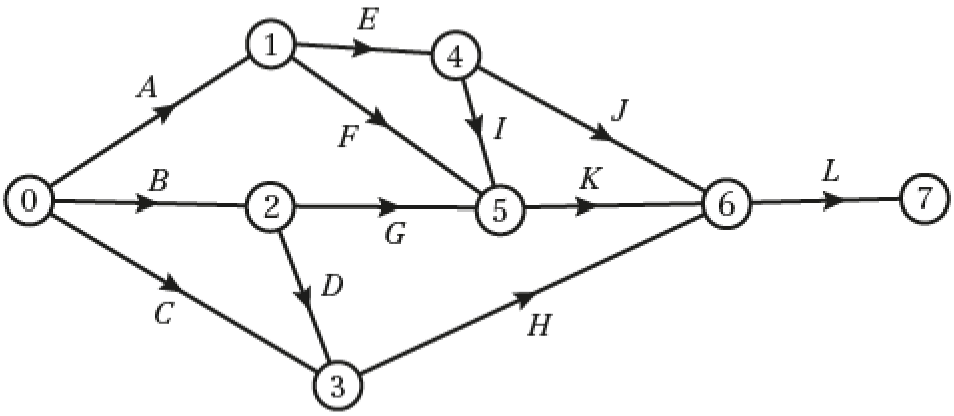
b Visible use of objective line method – objective line drawn (e.g. from (2, 0) to (0, 4)) or all 5 points tested.
vertex testing

$$\left[(0,\ 0) \to 0; (2,\ 8) \to 2.4; (4,\ 8) \to 3.2; (6.5,\ 5.5) \to 3.7; (6.5,\ 0) \to 2.6\right]$$

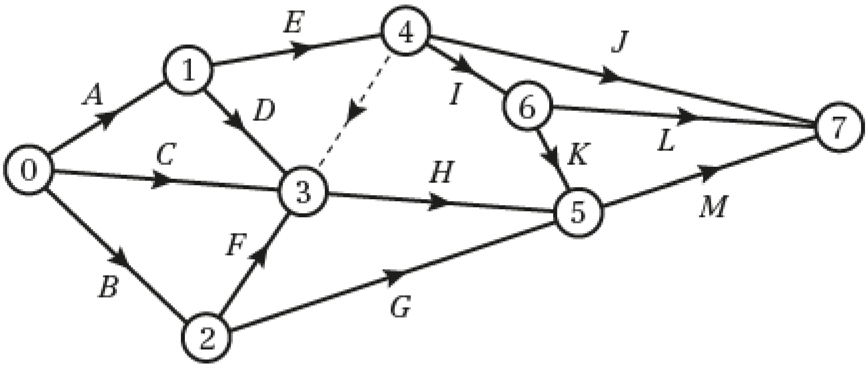
Optimal point is $(6.5,\ 5.5) \Rightarrow 6\,500$ type X and 5500 type Y

c $P = 0.4(6500) + 0.2(5500) = £3700$

22

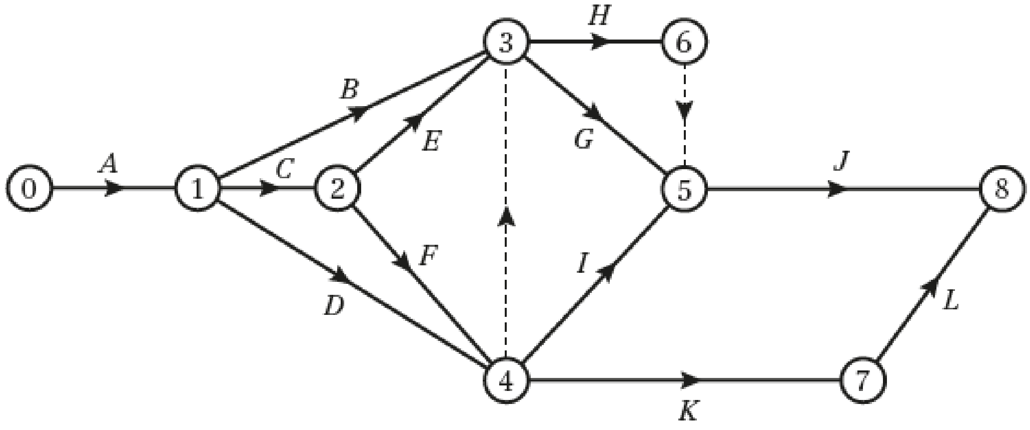


23 a



b Here we have that I and J depend only on E, whereas H depends on C, D, E and F. Hence we need separate nodes with a dummy.

24 a



b *D* will only be critical if it lies on the longest path

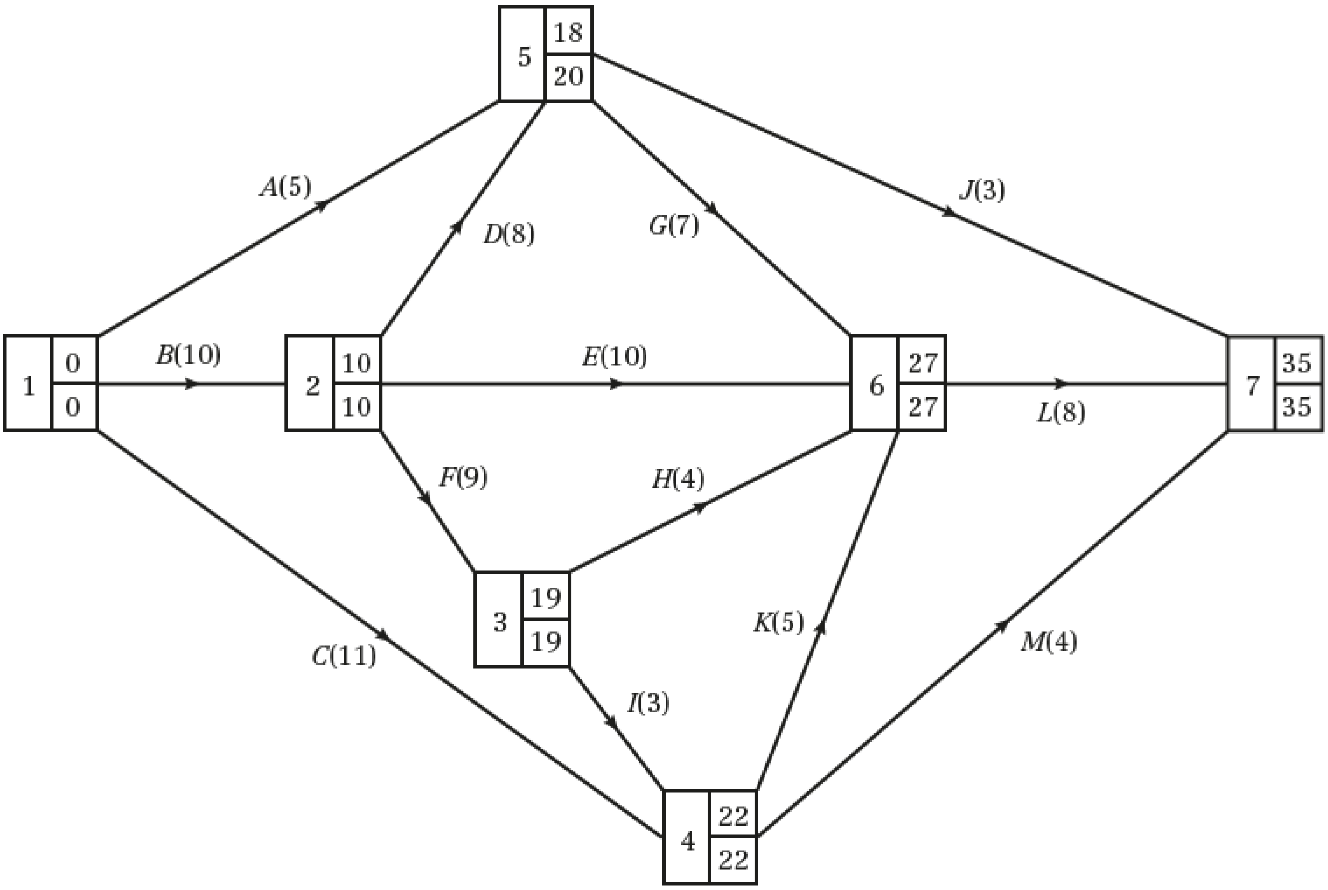
Path *A* to *G*	Length
ABEG	14
ACFG	15
ACDEG	$13 + x$

So we need $13 + x$ to be the longest, or equal longest

$13 + x \geqslant 15$

$x \geqslant 2$

25 a



25 b Total float on $A = 20 - 0 - 5 = 15$
Total float on $B = 10 - 0 - 10 = 0$
Total float on $C = 22 - 0 - 11 = 11$
Total float on $D = 20 - 10 - 8 = 2$
Total float on $E = 27 - 10 - 10 = 7$
Total float on $F = 19 - 10 - 9 = 0$
Total float on $G = 27 - 18 - 7 = 2$
Total float on $H = 27 - 19 - 4 = 4$
Total float on $I = 22 - 19 - 3 = 0$
Total float on $J = 35 - 18 - 3 = 14$
Total float on $K = 27 - 22 - 5 = 0$
Total float on $L = 35 - 27 - 8 = 0$
Total float on $M = 35 - 22 - 4 = 9$

c Critical activities: B, F, I, K and L
length of critical path is 35 days

d New critical path is $B - F - H - L$
length of new critical path is 36 days

26 a $x = 0$
$y = 7$ [latest out of $(3 + 2)$ and $(5 + 2)$]
$z = 9$ [Earliest out of $(13 - 4)$ and $(19 - 7)$ and $(16 - 2)$]

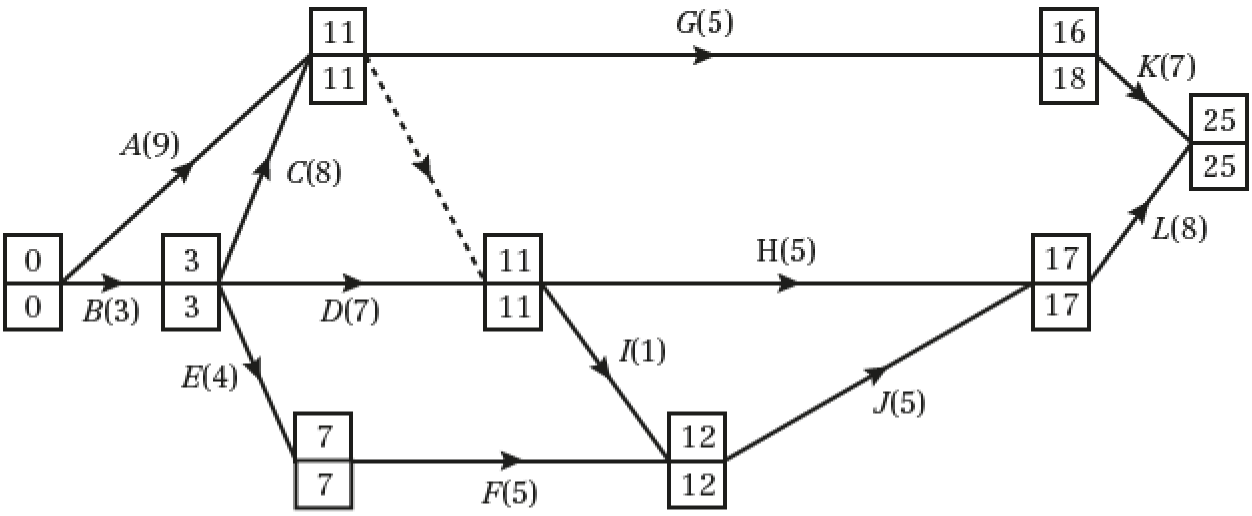
b Length is 22
Critical activities: B, D, E and L

c **i** Total float on $N = 22 - 14 - 3 = 5$

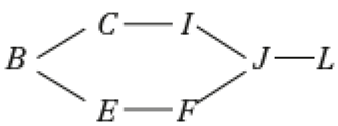
ii Total float on $H = 16 - 5 - 3 = 8$

27 a For example, it shows dependence but it is not an activity. G depends on A and C *only* but H and I depend on A, C and D.

b



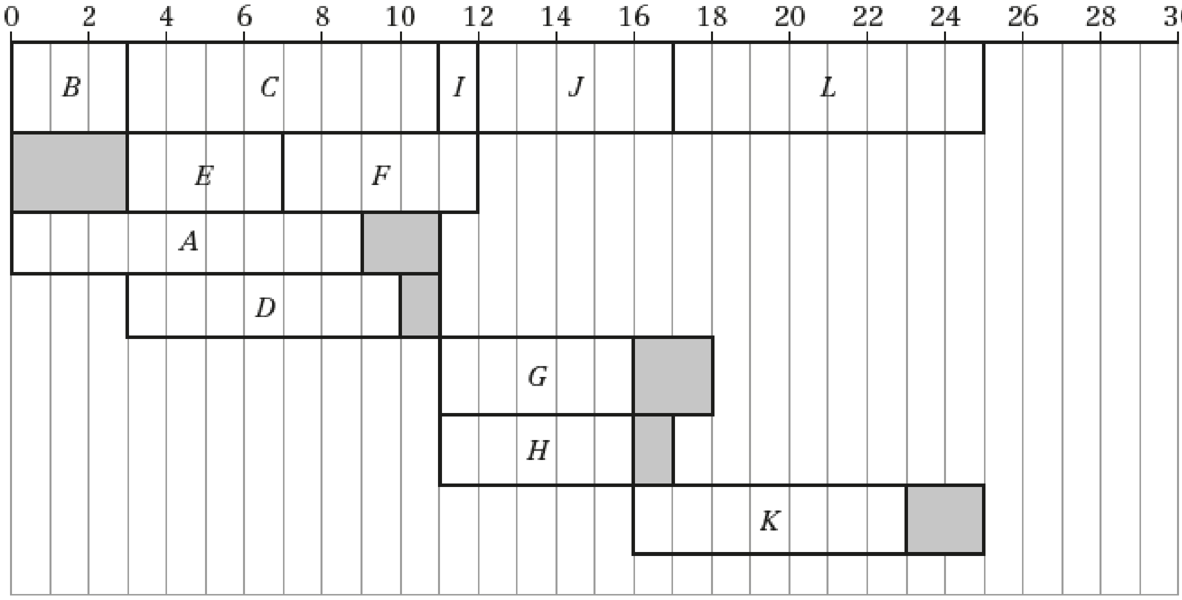
c



So B, C, E, F, I, J and L

d Total float on $A = 11 - 0 - 9 = 2$
Total float on $D = 11 - 3 - 7 = 1$
Total float on $G = 18 - 11 - 5 = 2$
Total float on $H = 17 - 11 - 5 = 1$
Total float on $K = 25 - 16 - 7 = 2$

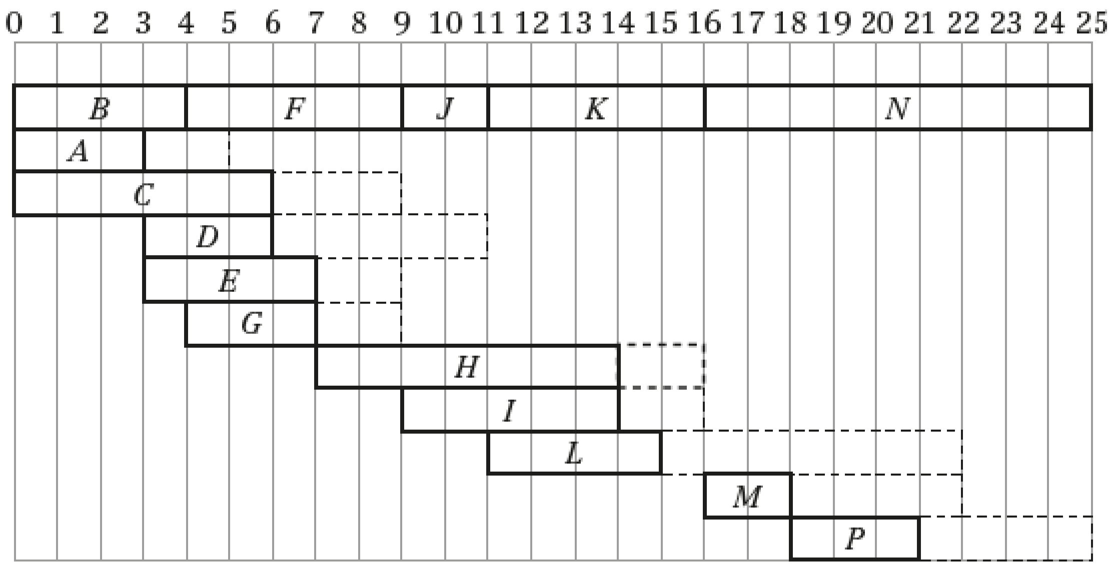
27 e



28 a Critical activities are B, F, J, K and N length of critical path is 25 hours
I is not critical.

b Total float on $A = 5 – 0 – 3 = 2$
Total float on $C = 9 – 0 – 6 = 3$
Total float on $D = 11 – 3 – 3 = 5$
Total float on $E = 9 – 3 – 4 = 2$
Total float on $G = 9 – 4 – 3 = 2$
Total float on $H = 16 – 7 – 7 = 2$
Total float on $I = 16 – 9 – 5 = 2$
Total float on $L = 22 –11 – 4 = 7$
Total float on $M = 22 – 16 – 2 = 4$
Total float on $P = 25 – 18 – 3 = 4$

c



d Look at 6.5 in the chart in **c**: F, E and G

Challenge

1 a $9\frac{1}{2}x - 26$

b The only vertices of odd order are B and C, we have to repeat the shortest path between B and C.
If $x \geqslant 9$ the shortest path is BC (direct)
Weight of network + BC = 100
$\left(9\frac{1}{2}x - 26\right) + x = 100 \Rightarrow x = 12$
If $x < 9$ the shortest path is BAC of length $2x - 9$
$\left(9\frac{1}{2}x - 26\right) + 2x - 9 = 100 \Rightarrow x = 11\frac{17}{23} \geqslant 9$
so inconsistent and hence $BA + AC \geqslant BC$

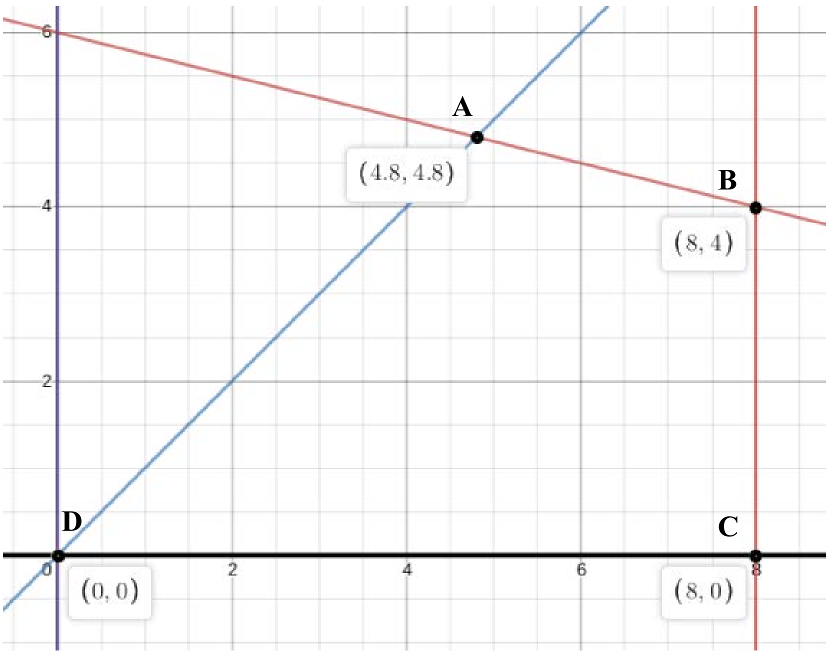
c The only vertices of odd order are B and C, we have to repeat the shortest path between B and C.
If $x \geqslant 9$ the shortest path is BC (direct).
Weight of network + BC = 100
$(9\frac{1}{2}x - 26) + x = 100 \Rightarrow x = 12$

If $x < 9$ the shortest path is BAC of length $2x - 9$
$(9\frac{1}{2}x - 26) + 2x - 9 = 100 \Rightarrow x = 11\frac{17}{23} \geqslant 9$ so inconsistent and hence $x = 12$

2 a Minimum spanning tree = 751
Initial upper bound = 2 × 751 = 1502
Taking shortcut AH saves 120 + 131 – 144 = 107
Tour length = 1395
Tour route: $ABACAEHFGDGFHA$

b Delete G and use Prim's algorithm starting at A
RMST = 672
Lower bound by deleting G = 672 + 144 + 155 = 971
Route is not a tour, so does not give an optimal solution.

3 a



$P_A = 4.8 + (5 \times 4.8) = 28.8$

$P_B = 8 + (5 \times 4) = 28$

$P_C = 8 + (5 \times 0) = 8$

$P_D = 0 + (0 \times 0) = 0$

Therefore, optimised when
x = 4.8, y = 4.8

3 b Test nearby integer points.
$P_1 = 4 + (5 \times 4) = 24$, $(x = 4, y = 4)$
$P_2 = 5 + (5 \times 4) = 25$, $(x = 5, y = 4)$
$P_3 = 6 + (5 \times 4) = 26$, $(x = 6, y = 4)$

This shows that (6,4) is the most optimal integer point.

c In this case, the gradient of the objective line makes the result sensitive to changes in y. Thus, as the non-integer solution is found at a value close to an integer value of y, and the nearest integer value is in the forbidden region, it takes a large change in x to offset the change in y.