Review Exercise 1

1 a

a	*b*	*c*	*d*	*e*	*f*	*f* = 0?
645	255	253	2	510	135	No
255	135	1.89	1	135	120	No
135	120	1.13	1	120	15	No
120	15	8	8	120	0	Yes

answer is 15

b The first row would be

255 645 0.40 0 0 255 No

but the second row would then be the same as the first row in the table above. So in effect this new first line would just be an additional row at the start of the solution.

c Finds the Highest Common Factor of *a* and *b*.

2 a Total length = 10 + 15 + 55 + 40 + 75 + 25 + 55 + 60 + 55 = 390 cm, so at least 4 planks are needed

b Bin 1: 10 + 15 + 55
Bin 2: 40 + 25
Bin 3: 75
Bin 4: 55
Bin 5: 60
Bin 6: 55
This uses 6 planks.

c By inspection,
Bin 1: 60+ 40 = 100
Bin 2: 75 + 25 = 100
Bin 3: 55 + 10 + 15 = 80
Bin 4: 55
Bin 5: 55
This uses 5 planks.

d There are 5 lengths over 50 cm, so none of these can be paired together. Therefore, minimum of 5 lengths are required.

3 a Bubbling left to right

55 80 25 84 25 34 17 75 3 5	$55 < 80$ so swap
80 55 25 84 25 34 17 75 3 5	$55 > 25$ so leave
80 55 84 25 25 34 17 75 3 5	$25 < 84$ so swap
80 55 84 25 25 34 17 75 3 5	$25 \nless 25$ so leave
80 55 84 25 34 25 17 75 3 5	$25 < 34$ so swap
80 55 84 25 34 25 17 75 3 5	$25 > 17$ so leave
80 55 84 25 34 25 75 17 3 5	$17 < 75$ so swap
80 55 84 25 34 25 75 17 3 5	$17 > 3$ so leave
80 55 84 25 34 25 75 17 5 3	$3 < 5$ so swap

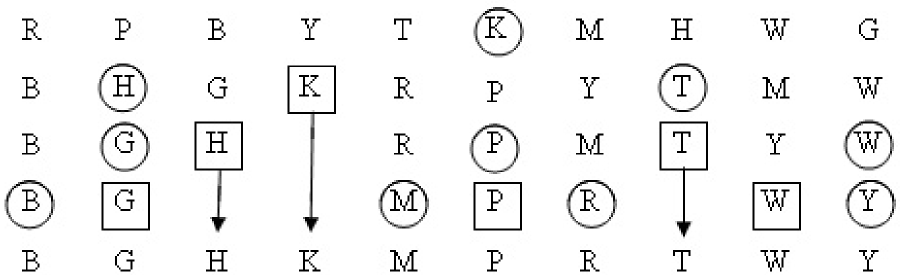
After 1st pass: 80 55 84 25 34 25 75 17 5 3
After 2nd pass: 80 84 55 34 25 75 25 17 5 3
After 3rd pass: 84 80 55 34 75 25 25 17 5 3
After 4th pass: 84 80 55 75 34 25 25 17 5 3
After 5th pass: 84 80 75 55 34 25 25 17 5 3
After 6th pass: 84 80 75 55 34 25 25 17 5 3
No swap in 6th pass, so the list is in order.

b $\frac{55+80+25+84+25+34+17+75+3+5}{100} = 4.03$ so 5 bins are needed.

c Using numbers sorted in descending order,
Bin 1: 84 + 5 + 3
Bin 2: 80 + 17
Bin 3: 75 + 25
Bin 4: 55 + 34
Bin 5: 25

4 a After one iteration we have	45	37	18	46	56	79	90	81	51
b After one pass we have	56	45	79	46	37	90	81	51	18

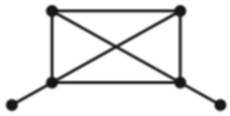
5 For example,



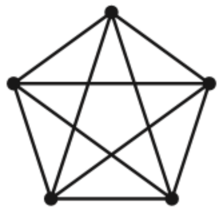
list is in order

6 a Since the graph is simple, there are no loops, so each of the degree-5 vertices must be joined to each of the other vertices. This means that each of the other vertices has degree at least 2.

b



7 a

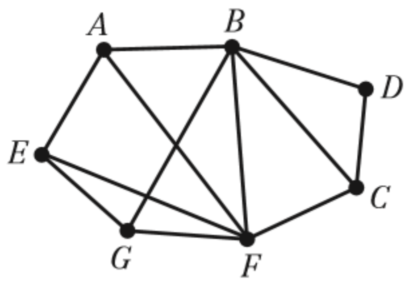


b 4: this includes all the vertices of the original graph and is also a tree.

c 5: cycle that includes every vertex.

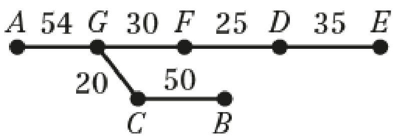
8 a A Hamiltonian cycle is a cycle that includes every vertex.

b *ABDCFGEA*



9 a *GC*, *FD*, *GF*, reject *CD*, *ED*, reject *EF*, *BC*, *AG*, reject *AB*.

b



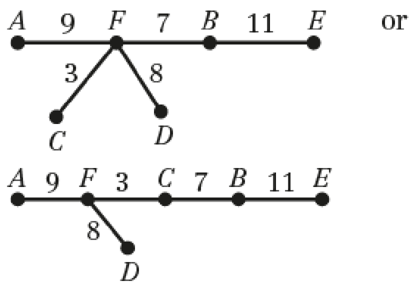
Cost = (20 + 25 + 30 + 35 + 50 + 54)×1000 = £214 000

10 a i Method:

- Start at *A* and use this to start the tree.
- Choose the shortest edge that connects a vertex already in the tree to a vertex not yet in the tree. Add it to the tree.
- Continue adding edges until all vertices are in the tree.

$$AF,\ FC\left\{\begin{matrix} FB \\ \text{or} \\ BC \end{matrix}\right\},FD,\ EB$$

10 a ii Total length of spanning tree = 9+3+7+8+11 = 38



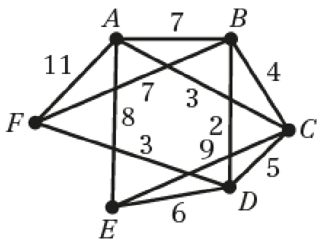
In a tree the number of edges is always one less than the number of nodes.

iii The tree is not unique, there are 2 of them (see above).

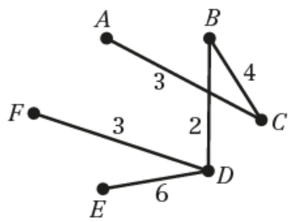
b i number of edges = 7 – 1 = 6

ii number of vertices = $n + 1$

11 a



b $BD, \begin{Bmatrix} AC \\ DF \end{Bmatrix}, BC$, reject CD, DE. Length of tree = 2 + 3 + 3 + 4 + 6 = 18 km



c *DB*, *DF*, *BC*, *CA*, *DE*

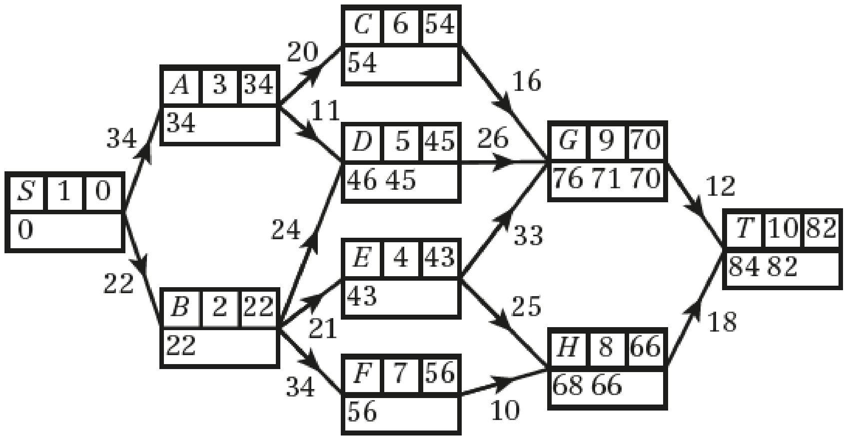
12 a In Prim's algorithm, the starting point can be any node, whereas Kruskal's algorithm starts from the arc of least weight. In Prim's algorithm, each new node and arc is added to the existing tree as it builds, whereas in applying Kruskal's algorithm, the arcs are selected according to their weight and may not be connected until the end.

b i Choose *G* to start the tree.
Add the arc of the least weight, *GH,* to the tree.
Consider arcs linking *G* to another vertex of least weight to the tree.
Continue to select an arc of least weight that joins a vertex already in the tree to a vertex not yet in the tree until all the vertices are connected: *GH, GI, HF, FD, DA, AB, AC, DE*

ii By inspection, order the arcs into ascending order of weight.
Select the arc of least weight to start the tree.
Consider the next arc of least weight, if it would form a cycle with the arcs already selected, reject it.
Continue to select an arc of least weight until all vertices are connected: *GH*, *AB*, *AC*, *AD*, reject *BD*, *DF*, *GI*, reject *BC*, *FH*, reject *DG, DE*

c Weight is 6 + 7 + 8 + 9 + 10 + 10 + 11 + 15 = 76

13 a



Route: *SACGT* length: 82

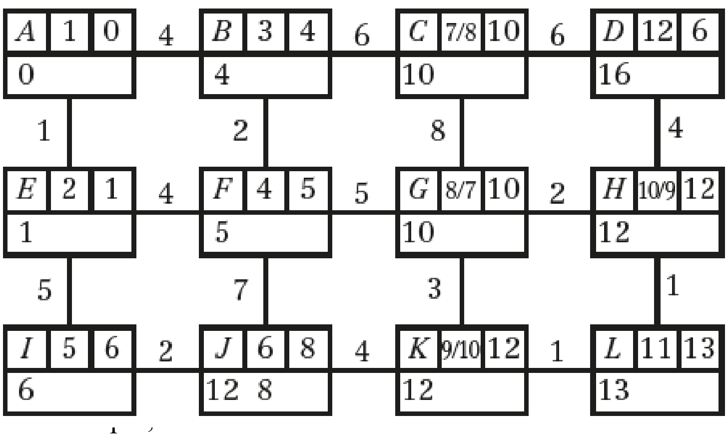
b For example

82 – 12 = 70 *GT*
70 – 16 = 54 *CG*
54 – 20 = 34 *AC*
34 – 34 = 0 *SA*

c Shortest route from *S* to *H* + *HT*

SBFHT length: 84

14 a



13 – 1 =12	*HL*	or	13 – 1 = 12	*KL*
12 – 2 = 10	*GH*		12 – 4 = 8	*JK*
10 – 5 = 5	*FG*		8 – 2 = 6	*IJ*
5 – 4 = 1	*EF*		6 – 5 = 1	*EI*
1 – 1 = 0	*AE*		1 – 1 = 0	*AE*

Shortest path is $\left\{ \begin{matrix} AEFGHL \\ AEIJKL \end{matrix} \right\}$ length 13

b See the two equal length paths given above in part **a**.

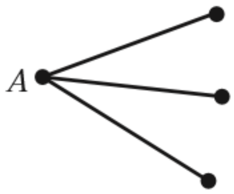
Challenge

1 a Let G be any finite simple graph with more than one vertex and with number of vertices $n \geqslant 2$. The maximal degree of any vertex in G is $\leqslant n - 1$. Also, if our graph G is not connected, then the maximal degree is $< n - 1$.
Case 1: Assume that G is connected. We cannot have a vertex of degree 0 in G, so the set of vertex degrees is a subset of $S = \{1, 2, \ldots, n - 1\}$. Since the graph G has n vertices and there are $n - 1$ possible degree options, there must be two vertices of the same degree in G.
Case 2: Assume that G is not connected. G has no vertex of degree $n - 1$, so the set of vertex degrees is a subset of $S' = \{1, 2, \ldots, n - 2\}$. Again, we have n vertices and $n - 1$ possible degree options, so there must be two vertices of the same degree in G.

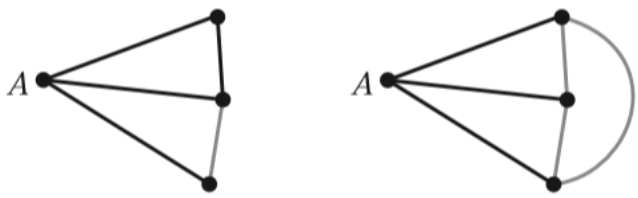
b i By inspection, possible sets are:
Blue: *ABD*, *ACD*
Red: *BCF*, *DEF*

ii For K_6 any vertex will have a valency of 5, an edge to each of the other points.

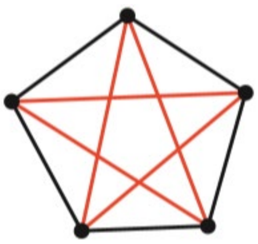
With 5 lines there must be at least three of one colour so there are four points connected by the same colour.



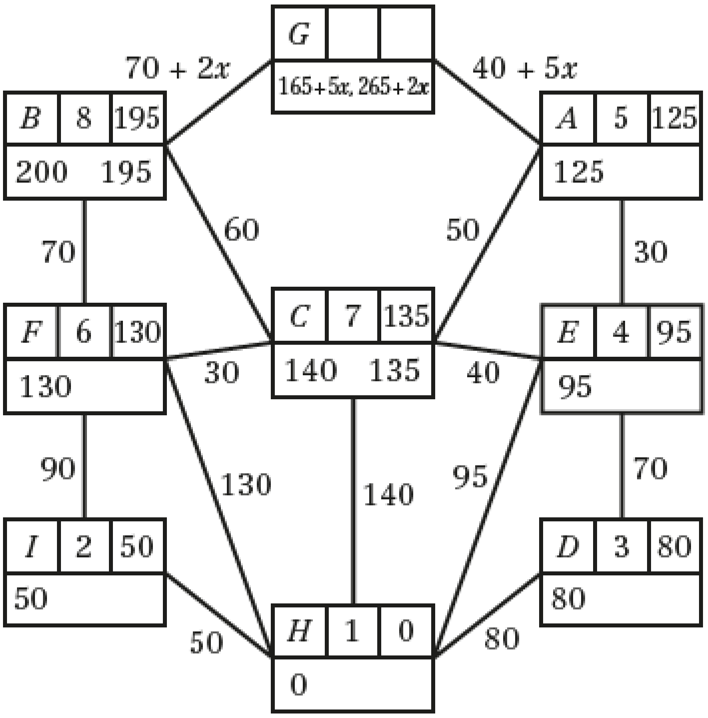
For the three lines connecting the new points, if one is the original colour then a set of three vertices is made with the original colour. If none are the original colour then the three vertices make a set of three themselves.



iii



2 a



Via A: *HEAG* length $165 + 5x$
Via B: *HECBG* length $265 + 2x$

b $165+5x=265+2x \Rightarrow x=33\frac{1}{3}$

So range is $0 \leqslant x \leqslant 33\frac{1}{3}$