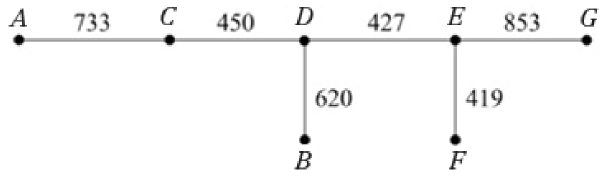
## Chapter Review 5

**1** **a** Either Kruskal: *EF, DE, CD, BD, AC, EG* or Prim (e.g.): *AC, CD, DE, EF, BD, EG*




**b** $2 \times 3502 = 7004$

**c** For example use *AB* and *DG*
Route *ACDEFEGDBA* length 6005

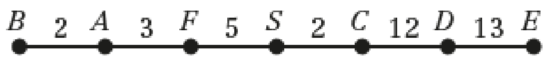
**2** **a**

| | *A* | *B* | *C* | *D* | *E* |
|---|---|---|---|---|---|
| *A* | – | 7 | 13 | 4 | 3 |
| *B* | 7 | – | 17 | 7 | 10 |
| *C* | 13 | 17 | – | 10 | 13 |
| *D* | 4 | 7 | 10 | – | 5 |
| *E* | 3 | 10 | 13 | 5 | – |

**b** $A_3E_5D_7B_{17}C_{13}A = 45$

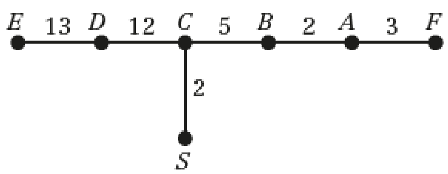
**c** *AEDBDCA* (*BC* is not on the original network.)

**3** **a** *SC SF FA AB CD DE* – tree 1




and

*SC CB BA AF CD DE* – tree 2




**b** Weight of each tree is 37
So initial upper bound is $2 \times 37 = 74$

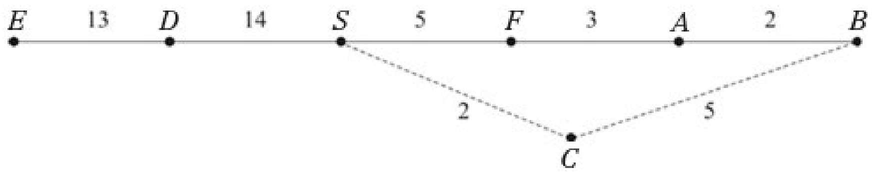
**c** From tree 1
Use *BE* as a shortcut
(Route is *SCDEBAFS*) length 56
From tree 2
Use *EF* as a shortcut
(Route is *SCBAFEDCS*) length 53

**4 d** $C_2S_5F_3A_2B_{17}D_{13}E_{21}C = 63$
$D_{12}C_2S_5F_3A_2B_{19}E_{13}D = 56$

**e** The better upper bound is 53 since it is smaller.

**f** The route is *SCBAFEDCS*

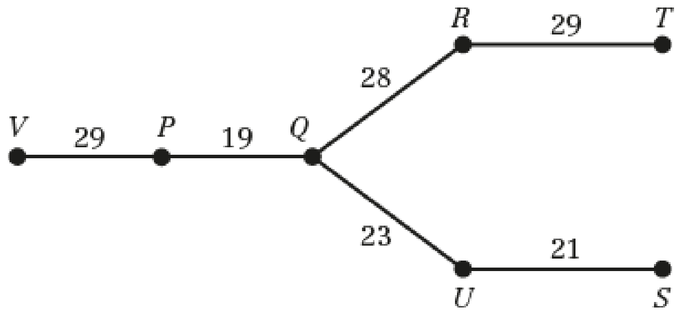
**g**




Weight of residual minimum spanning tree = 37
Two least arcs from *C* are *CS* and *CB*
Lower bound = 37 + 2 + 5 = 44

**4 a** In the classical problem each vertex must be visited exactly once before returning to the start.
In the practical problem each vertex must be visited at least once before returning to the start.

**b**

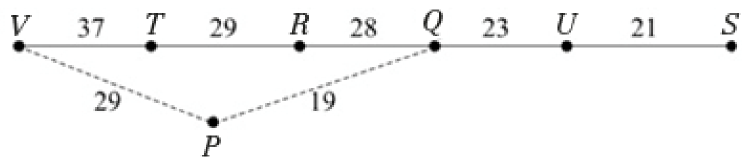



Order of arcs: *PQ*, *QU*, *US*, *QR*, $\begin{Bmatrix} TR \\ VP \end{Bmatrix}$

**c** Use *VT* and *QS* as shortcuts giving a length of 213
(Route *PQUSQRTVP*)

**d** $P_{19}Q_{23}U_{21}S_{51}R_{29}T_{37}V_{29}P = 209$

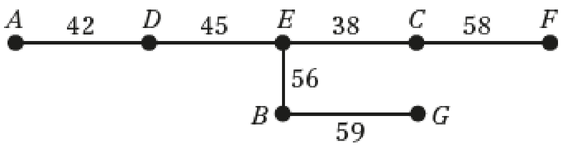
**e**




Weight of residual minimum spanning tree = 138
Two least arcs *PQ* and *PV*
Lower bound = 138 + 19 + 29 = 186

**f** $186 < \text{optimal value} \leqslant 209$

**5 a**



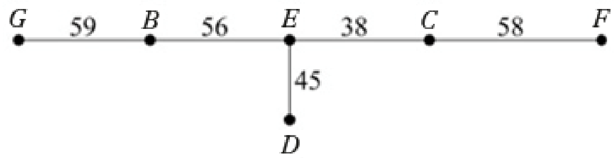

Order of arcs: *AD*, *DE*, *EC*, *EB*, *CF*, *BG*

**b** Initial upper bound = 2 × 298
= 596

**c** The minimum connector has been doubled and each arc in it repeated.

**d** Use *AE* and *GF* as shortcuts – length 427
(route is *ADEBGFCEA*)

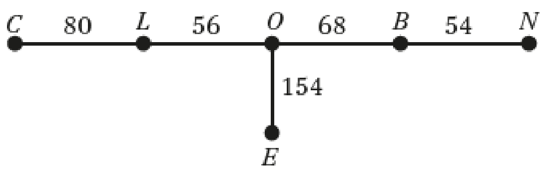
**e**




Weight of residual minimum spanning tree = 256
Two least arcs from *A* are *AD* (42) and *AE* (54)
Lower bound = 256 + 42 + 54 = 352 km

**f** The lower bound will give the optimal solution if it is a tour.
If the minimum spanning tree has no 'branches' – so the two end vertices have valency 1, and all other vertices have valency 2, then if the two least arcs are incident on the 2 vertices of valency 1 an optimal solution cannot be found.

**6 a**

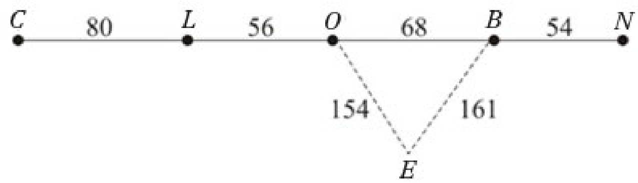



order of selection: *LO*, *OB*, *BN*, *LC*, *OE*

**b i** Initial upper bound = 2 × 412
= 824 miles

**ii** Use *NC* as a shortcut – length is 653
(Route is *LOEOBNCL*)

**c**




Weight of residual minimum spanning tree = 258
Two least arcs are *EO* and *EB*
Lower bound = 258 + 154 + 161 = 573

**7** **a** The nearest neighbour route is *AECGBDFA* of length $12 + 22 + 23 + 20 + 18 + x + 15 = 110 + x$
Hence, $140 = 110 + x \Rightarrow x = 30$.

**b** The nearest neighbour route from *B* is *BAECGFDB* of length $16 + 12 + 22 + 23 + 30 + x + 18 = 151$ miles.

**c** By using Prim's algorithm (table below) or otherwise we find the RMST of length
$16 + 21 + 17 + 12 + 15 = 81$

| | *A* | *B* | *C* | *D* | *E* | *F* |
|---|---|---|---|---|---|---|
| *A* | – | 16 | 21 | 17 | 12 | 15 |
| *B* | (16) | – | 24 | 18 | 30 | 26 |
| *C* | (21) | 24 | – | 31 | 22 | 35 |
| *D* | (17) | 18 | 31 | – | 28 | $x$ |
| *E* | (12) | 30 | 22 | 28 | – | 27 |
| *F* | (15) | 26 | 35 | $x$ | 27 | – |

Two shortest edges from *G* to the reduced graph are *GA* and *GB* of lengths 19 and 20, respectively. Hence, we have a lower bound of $81 + 19 + 20 = 120$ miles.

**d** Using the upper bound of 140 given in the question we have $120 <$ optimal value $\leqslant 140$.

## Challenge

**a** Using the nearest neighbour algorithm

Starting at *A*

| | 1 | 2 | 7 | 6 | 5 | 4 | 3 |
|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 14 | 11 |
| *B* | (4) | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | (9) | 16 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | (9) | 16 | 16 |
| *E* | 17 | 14 | 16 | 9 | – | (10) | 18 |
| *F* | 14 | 17 | 20 | 16 | 10 | – | (10) |
| *G* | 11 | (8) | 15 | 16 | 18 | 10 | – |

The first arc is *AB* (4).
The second arc is *BG* (8).
The third arc is *GF* (10).
The fourth arc is *FE*(10).
The fifth arc is *ED* (9).
The sixth arc is *DC* (9).
The shortest route is *ABGFEDC* which has weight:
$4 + 8 + 10 + 10 + 9 + 9 = 50$
*CA* has weight 8 so *ABGFEDCA* has weight 58

Starting at *B*

| | 2 | 1 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 14 | 11 |
| *B* | 4 | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | 9 | 16 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | 9 | 16 | 16 |
| *E* | 17 | 14 | 16 | 9 | – | 10 | 18 |
| *F* | 14 | 17 | 20 | 16 | 10 | – | 10 |
| *G* | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *BA* (4).
The second arc is *AC* (8).
The third arc is *CD* (9).
The fourth arc is *DE* (9).
The fifth arc is *EF* (10).
The sixth arc is *FG* (10).
The shortest route is *BACDEFG* which has weight:
4 + 8 + 9 + 9 + 10 + 10 = 50
*GB* has weight 8 so *BACDEFGB* has weight 58

Starting at *C*

| | 2 | 3 | 1 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 14 | 11 |
| *B* | 4 | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | 9 | 16 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | 9 | 16 | 16 |
| *E* | 17 | 14 | 16 | 9 | – | 10 | 18 |
| *F* | 14 | 17 | 20 | 16 | 10 | – | 10 |
| *G* | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *CA* (8).
The second arc is *AB* (4).
The third arc is *BG* (8).
The fourth arc is *GF* (10).
The fifth arc is *FE* (10).
The sixth arc is *ED* (9).
The shortest route is *CABGFED* which has weight:
8 + 4 + 8 + 10 + 10 + 9 = 49
*CD* has weight 9 so *CABGFED* has weight 58

Starting at *D*

| | 3 | 4 | 2 | 1 | 7 | 6 | 5 |
|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 14 | 11 |
| *B* | 4 | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | 9 | 16 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | 9 | 16 | 16 |
| *E* | 17 | 14 | 16 | 9 | – | 10 | 18 |
| *F* | 14 | 17 | 20 | 16 | 10 | – | 10 |
| *G* | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *DC* (9).
The second arc is *CA* (8).
The third arc is *AB* (4).
The fourth arc is *BG* (8).
The fifth arc is *GF* (10).
The sixth arc is *FE* (10).
The shortest route is *DCABGFE* which has weight:
9 + 8 + 4 + 8 + 10 + 10 = 49
*ED* has weight 9 so *DCABGFED* has weight 58

Starting at $E$

| | 4 | 5 | 3 | 2 | 1 | 7 | 6 |
|---|---|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| $A$ | – | 4 | 8 | 16 | 17 | 14 | 11 |
| $B$ | 4 | – | 11 | 15 | 14 | 17 | 8 |
| $C$ | 8 | 11 | – | 9 | 16 | 20 | 15 |
| $D$ | 16 | 15 | 9 | – | 9 | 16 | 16 |
| $E$ | 17 | 14 | 16 | 9 | – | 10 | 18 |
| $F$ | 14 | 17 | 20 | 16 | 10 | – | 10 |
| $G$ | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is $ED$ (9).
The second arc is $DC$ (9).
The third arc is $CA$ (8).
The fourth arc is $AB$ (4).
The fifth arc is $BG$ (8).
The sixth arc is $GF$ (10).
The shortest route $EDCABGF$ is which has weight:
$9 + 9 + 8 + 4 + 8 + 10 = 48$
$FE$ has weight 10 so $EDCABGFE$ has weight 58

Starting at $F$

| | 5 | 6 | 4 | 3 | 2 | 1 | 7 |
|---|---|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ |
| $A$ | – | 4 | 8 | 16 | 17 | 14 | 11 |
| $B$ | 4 | – | 11 | 15 | 14 | 17 | 8 |
| $C$ | 8 | 11 | – | 9 | 16 | 20 | 15 |
| $D$ | 16 | 15 | 9 | – | 9 | 16 | 16 |
| $E$ | 17 | 14 | 16 | 9 | – | 10 | 18 |
| $F$ | 14 | 17 | 20 | 16 | 10 | – | 10 |
| $G$ | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *FE* (10).
The second arc is *ED* (9).
The third arc is *DC* (9).
The fourth arc is *CA* (8).
The fifth arc is *CB* (4).
The sixth arc is *BG* (8).
The shortest route is *FEDCABG* which has weight:
10 + 9 + 9 + 8 + 4 + 8 = 48
*GF* has weight 10 so *FEDCABGF* has weight 58

Starting at *G*

| | 3 | 2 | 4 | 5 | 6 | 7 | 1 |
|---|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 14 | 11 |
| *B* | 4 | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | 9 | 16 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | 9 | 16 | 16 |
| *E* | 17 | 14 | 16 | 9 | – | 10 | 18 |
| *F* | 14 | 17 | 20 | 16 | 10 | – | 10 |
| *G* | 11 | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *GB* (8).
The second arc is *BA* (4).
The third arc is *AC* (8).
The fourth arc is *CD* (9).
The fifth arc is *DE* (9).
The sixth arc is *EF* (10).
The shortest route is *GBACDEF* which has weight:
8 + 4 + 8 + 9 + 9 + 10 = 48
*FG* has weight 10 so *GBACDEFG* has weight 58
All routes have weight 58.
So the best upper bound is 58.

Removing *A*

| | 1 | 6 | 5 | 4 | 3 | 2 |
|---|---|---|---|---|---|---|
| | *B* | *C* | *D* | *E* | *F* | *G* |
| *B* | – | 11 | 15 | 14 | 17 | 8 |
| *C* | 11 | – | 9 | 16 | 20 | 15 |
| *D* | 15 | 9 | – | 9 | 16 | 16 |
| *E* | 14 | 16 | 9 | – | 10 | 18 |
| *F* | 17 | 20 | 16 | 10 | – | 10 |
| *G* | 8 | 15 | 16 | 18 | 10 | – |

The first arc is *BG* (8).
The second arc is *GF* (10).
The third arc is *FE* (10).
The fourth arc is *ED* (9).
The fifth arc is *DC* (9).
Lower bound = weight of RMST + weights of two least arcs from *A*
= 46 + 4 + 8
= 58

Removing *B*

| | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| | *A* | *C* | *D* | *E* | *F* | *G* |
| *A* | – | 8 | 16 | 17 | 14 | 11 |
| *C* | 8 | – | 9 | 16 | 20 | 15 |
| *D* | 16 | 9 | – | 9 | 16 | 16 |
| *E* | 17 | 16 | 9 | – | 10 | 18 |
| *F* | 14 | 20 | 16 | 10 | – | 10 |
| *G* | 11 | 15 | 16 | 18 | 10 | – |

The first arc is *AC* (8).
The second arc is *CD* (9).
The third arc is *DE* (9).
The fourth arc is *EF* (10).
The fifth arc is *FG* (10).
Lower bound = weight of RMST + weights of two least arcs from *B*
= 46 + 4 + 8
= 58

Removing *C*

| | 1 | 2 | 6 | 5 | 4 | 3 |
|---|---|---|---|---|---|---|
| | *A* | *B* | *D* | *E* | *F* | *G* |
| *A* | – | 4 | 16 | 17 | 14 | 11 |
| *B* | 4 | – | 15 | 14 | 17 | 8 |
| *D* | 16 | 15 | – | 9 | 16 | 16 |
| *E* | 17 | 14 | 9 | – | 10 | 18 |
| *F* | 14 | 17 | 16 | 10 | – | 10 |
| *G* | 11 | 8 | 16 | 18 | 10 | – |

The first arc is *AB* (4).
The second arc is *BG* (8).
The third arc is *GF* (10).
The fourth arc is *FE* (10).
The fifth arc is *ED* (9).
Lower bound = weight of RMST + weights of two least arcs from *C*
= 41 + 8 + 9
= 58

Removing *D*

| | 1 | 2 | 3 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *E* | *F* | *G* |
| *A* | – | 4 | 8 | 17 | 14 | 11 |
| *B* | 4 | – | 11 | 14 | 17 | 8 |
| *C* | 8 | 11 | – | 16 | 20 | 15 |
| *E* | 17 | 14 | 16 | – | 10 | 18 |
| *F* | 14 | 17 | 20 | 10 | – | 10 |
| *G* | 11 | 8 | 15 | 18 | 10 | |

The first arc is *AB* (4).
The second arc is *BC* (8).
The third arc is *BG* (8).
The fourth arc is *GF* (10).
The fifth arc is *FE* (10).
Lower bound = weight of RMST + weights of two least arcs from *D*
= 40 + 9 + 9
= 58

Removing *E*

| | 1 | 2 | 4 | 5 | 6 | 3 |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *F* | *G* |
| *A* | – | 4 | 8 | 16 | 14 | 11 |
| *B* | 4 | – | 11 | 15 | 17 | 8 |
| *C* | 8 | 11 | – | 9 | 20 | 15 |
| *D* | 16 | 15 | 9 | – | 16 | 16 |
| *F* | 14 | 17 | 20 | 16 | – | 10 |
| *G* | 11 | 8 | 15 | 16 | 10 | – |

The first arc is *AB* (4).
The second arc is *BG* (8).
The third arc is *AC* (8).
The fourth arc is *CD* (9).
The fifth arc is *GF* (10).
Lower bound = weight of RMST + weights of two least arcs from *E*
= 39 + 9 + 10
= 58

Removing *F*

| | 1 | 2 | 4 | 5 | 6 | 3 |
|---|---|---|---|---|---|---|
| | *A* | *B* | *C* | *D* | *E* | *G* |
| *A* | – | 4 | 8 | 16 | 17 | 11 |
| *B* | 4 | – | 11 | 15 | 14 | 8 |
| *C* | 8 | 11 | – | 9 | 16 | 15 |
| *D* | 16 | 15 | 9 | – | 9 | 16 |
| *E* | 17 | 14 | 16 | 9 | | 18 |
| *G* | 11 | 8 | 15 | 16 | 18 | – |

The first arc is *AB* (4).
The second arc is *BG* (8).
The third arc is *AC* (8).
The fourth arc is *CD* (9).
The fifth arc is *DE* (9).
Lower bound = weight of RMST + weights of two least arcs from *F*
= 38 + 10 + 14
= 62

Removing $G$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ |
| $A$ | – | 4 | 8 | 16 | 17 | 14 |
| $B$ | 4 | – | 11 | 15 | 14 | 17 |
| $C$ | 8 | 11 | – | 9 | 16 | 20 |
| $D$ | 16 | 15 | 9 | – | 9 | 16 |
| $E$ | 17 | 14 | 16 | 9 | – | 10 |
| $F$ | 14 | 17 | 20 | 16 | 10 | – |

The first arc is $AB$ (4).
The second arc is $AC$ (8).
The third arc is $CD$ (9).
The fourth arc is $DE$ (9).
The fifth arc is $EF$ (10).
Lower bound = weight of RMST + weights of two least arcs from $G$
= 40 + 8 + 11
= 59
There is a lower bound of 58 and an upper bound of 58, therefore there is an optimal solution of weight 58.
The upper bound can be used: