⬡ **C4** 〉 **INTEGRATION**  *Answers - Worksheet P*

**1**  **a**  when $y = 3$, $x = \frac{1}{3}$

area $0 \le x \le \frac{1}{3} = 3 \times \frac{1}{3} = 1$

area $\frac{1}{3} < x \le 3 = \int_{\frac{1}{3}}^{3} \frac{1}{x}\ \mathrm{d}x$

$= [\ln | x |]_{\frac{1}{3}}^{3}$

$= \ln 3 - \ln \frac{1}{3} = \ln \frac{3}{\frac{1}{3}} = \ln 9$

total area $= 1 + \ln 9$

**b**  volume $0 \le x \le \frac{1}{3}$ = volume of cylinder

$= \pi \times 3^2 \times \frac{1}{3} = 3\pi$

volume $\frac{1}{3} < x \le 3 = \pi \int_{\frac{1}{3}}^{3} (\frac{1}{x})^2\ \mathrm{d}x$

$= \pi \int_{\frac{1}{3}}^{3} x^{-2}\ \mathrm{d}x = \pi[-x^{-1}]_{\frac{1}{3}}^{3}$

$= \pi[-\frac{1}{3} - (-3)] = \frac{8}{3}\pi$

total volume $= 3\pi + \frac{8}{3}\pi = \frac{17}{3}\pi$

**2**  **a**

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $x \sec (\frac{1}{3}x)$ | 0 | 0.507 | 1.058 | 1.709 | 2.545 | 3.718 | 5.552 | 8.901 | 17.004 |

**i**  $\approx \frac{1}{2} \times 2 \times [0 + 17.004 + 2(2.545)] = 22.1$ (3sf)

**ii**  $\approx \frac{1}{2} \times 1 \times [0 + 17.004 + 2(1.058 + 2.545 + 5.552)] = 17.7$ (3sf)

**iii**  $\approx \frac{1}{2} \times 0.5 \times [0 + 17.004 + 2(0.507 + 1.058 + 1.709 + 2.545 + 3.718 + 5.552 + 8.901)] = 16.2$ (3sf)

**b**  e.g.  halving the interval width leads to the estimate reducing by 4.4, then 1.5
further halving might lead to reductions of approximately 0.5 and 0.2 so $I \approx 15.5$

**3**  **a**  $-\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = k\theta$

$\int \frac{1}{\theta}\ \mathrm{d}\theta = \int -k\ \mathrm{d}t$

$\ln | \theta | = -kt + c$

$\theta = \mathrm{e}^{-kt + c} = \mathrm{e}^c \times \mathrm{e}^{-kt}$

$\therefore\ \theta = A\mathrm{e}^{-kt}$

$t = 0$, $\theta = 25 - 10 = 15$

$\therefore\ A = 15$

$\therefore\ \theta = 15\mathrm{e}^{-kt}$

**b**  $t = 30$, $\theta = 20 - 10 = 10$

$\therefore\ 10 = 15\mathrm{e}^{-30k}$

$k = -\frac{1}{30} \ln \frac{2}{3} = 0.0135$ (3sf)

**c**  $\theta = 15 - 10 = 5$

$\therefore\ 5 = 15\mathrm{e}^{-0.01352t}$

$t = -\frac{1}{0.01352} \ln \frac{1}{3}$

$= 81.3$ minutes (3sf)

**4**  **a**  $\sin^4 x \equiv (\sin^2 x)^2$

$\equiv [\frac{1}{2}(1 - \cos 2x)]^2$

$\equiv \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$\equiv \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}[\frac{1}{2}(1 + \cos 4x)]$

$\equiv \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$

$\therefore\ p = \frac{3}{8}$, $q = -\frac{1}{2}$, $r = \frac{1}{8}$

**b**  $= \int_{0}^{\frac{\pi}{2}} (\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x)\ \mathrm{d}x$

$= [\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x]_{0}^{\frac{\pi}{2}}$

$= (\frac{3}{16}\pi - 0 + 0) - (0 - 0 + 0)$

$= \frac{3}{16}\pi$

**5**  **a**  $\int y^{-3} \, dy = \int x \, dx$

$-\frac{1}{2} y^{-2} = \frac{1}{2} x^2 + c$

$y^{-2} = k - x^2$

$y^2 = \dfrac{1}{k - x^2}$

**b**  $y = \frac{1}{2}$ when $x = 1$

$\therefore \ \frac{1}{4} = \dfrac{1}{k-1}$

$k = 5$

$\therefore \ y^2 = \dfrac{1}{5 - x^2}$

**6**  **a**  $x = e^u \ \therefore \ u = \ln x, \ \dfrac{dx}{du} = e^u$

$\int \dfrac{2 + \ln x}{x^2} \, dx = \int \dfrac{2 + u}{(e^u)^2} \times e^u \, du$

$\qquad\qquad\qquad = \int (2 + u)e^{-u} \, du$

**b**  $x = e^u$

$x = 1 \ \Rightarrow \ u = 0$

$x = e \ \Rightarrow \ u = 1$

$\therefore \ \displaystyle\int_1^e \dfrac{2 + \ln x}{x^2} \, dx = \int_0^1 (2 + u)e^{-u} \, du$

$v = 2 + u, \ \dfrac{dv}{du} = 1; \ \dfrac{dw}{du} = e^{-u}, w = -e^{-u}$

$= [-(2 + u)e^{-u}]_0^1 + \displaystyle\int_0^1 e^{-u} \, du$

$= [-(2 + u)e^{-u} - e^{-u}]_0^1$

$= (-3e^{-1} - e^{-1}) - (-2 - 1)$

$= 3 - 4e^{-1}$

**7**  **a**  $x = 0 \ \Rightarrow \ \cos 2t = 0 \ \Rightarrow \ t = \frac{\pi}{4}$

$y = 0 \ \Rightarrow \ \tan t = 0 \ \Rightarrow \ t = 0$

$x = \cos 2t \ \therefore \ \dfrac{dx}{dt} = -2 \sin 2t$

shaded area $= \displaystyle\int_{\frac{\pi}{4}}^0 \tan t \times (-2 \sin 2t) \, dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} \dfrac{\sin t}{\cos t} \times 4 \sin t \cos t \, dt$

$= \displaystyle\int_0^{\frac{\pi}{4}} 4 \sin^2 t \, dt$

**b**  $= \displaystyle\int_0^{\frac{\pi}{4}} (2 - 2 \cos 2t) \, dt$

$= [2t - \sin 2t]_0^{\frac{\pi}{4}}$

$= (\frac{\pi}{2} - 1) - (0 - 0)$

$= \frac{\pi}{2} - 1$

**c**  **i**  $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

**ii**  $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$y^2 = \tan^2 t = \dfrac{\sin^2 t}{\cos^2 t} = \dfrac{\frac{1}{2}(1 - \cos 2t)}{\frac{1}{2}(1 + \cos 2t)}$

$\therefore \ y^2 = \dfrac{1 - x}{1 + x}$

**8**  **a**  $\dfrac{6 - 2x^2}{(x+1)^2 (x+3)} \equiv \dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

$6 - 2x^2 \equiv A(x+3) + B(x+1)(x+3) + C(x+1)^2$

$x = -1 \quad \Rightarrow \ 4 = 2A \qquad \Rightarrow \ A = 2$

$x = -3 \quad \Rightarrow \ -12 = 4C \ \Rightarrow \ C = -3$

coeffs $x^2 \Rightarrow \ -2 = B + C \Rightarrow B = 1$

**b**  $x = 0 \ \Rightarrow \ y = 2$

$f(x) = 2(x + 1)^{-2} + (x + 1)^{-1} - 3(x + 3)^{-1}$

$f'(x) = -4(x + 1)^{-3} - (x + 1)^{-2} + 3(x + 3)^{-2}$

$\therefore \ f'(0) = -4 - 1 + \frac{1}{3} = -\frac{14}{3}$

$\therefore \ y - 2 = -\frac{14}{3}(x - 0)$

$3y - 6 = -14x$

$14x + 3y = 6$

**c**  $= \displaystyle\int_0^1 \left( \dfrac{2}{(x+1)^2} + \dfrac{1}{x+1} - \dfrac{3}{x+3} \right) dx$

$= [-2(x + 1)^{-1} + \ln|x + 1| - 3 \ln|x + 3|]_0^1$

$= (-1 + \ln 2 - 3 \ln 4) - (-2 + 0 - 3 \ln 3)$

$= 1 - 5 \ln 2 + 3 \ln 3$