

C4 INTEGRATION

Answers - Worksheet O

$$1 \quad \mathbf{a} \quad \frac{1}{x^2-3x+2} \equiv \frac{A}{x-1} + \frac{B}{x-2}$$

$$1 \equiv A(x-2) + B(x-1)$$

$$x=1 \Rightarrow 1 = -A \Rightarrow A = -1$$

$$x=2 \quad \Rightarrow B = 1$$

$$\therefore \frac{1}{x^2-3x+2} \equiv \frac{1}{x-2} - \frac{1}{x-1}$$

$$\begin{aligned} \mathbf{b} \quad \int_3^4 \frac{1}{x^2-3x+2} dx &= \int_3^4 \left(\frac{1}{x-2} - \frac{1}{x-1} \right) dx \\ &= [\ln|x-2| - \ln|x-1|]_3^4 \\ &= (\ln 2 - \ln 3) - (0 - \ln 2) \\ &= 2 \ln 2 - \ln 3 = \ln \frac{2^2}{3} \\ &= \ln \frac{4}{3} \quad [a=4, b=3] \end{aligned}$$

$$3 \quad \mathbf{a} \quad \begin{array}{r} x+2 \\ x-1 \overline{) x^2 + x - 1} \\ \underline{x^2 - x} \\ 2x - 1 \\ \underline{2x - 2} \\ 1 \end{array}$$

quotient: $x+2$, remainder: 1

$$\begin{aligned} \mathbf{b} \quad \int \frac{x^2+x-1}{x-1} dx &= \int \left(x+2 + \frac{1}{x-1} \right) dx \\ &= \frac{1}{2}x^2 + 2x + \ln|x-1| + c \end{aligned}$$

$$5 \quad \mathbf{a} \quad u = \ln x, \frac{du}{dx} = \frac{1}{x}; \quad \frac{dv}{dx} = x, v = \frac{1}{2}x^2$$

$$\begin{aligned} \int x \ln x dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c \\ &= \frac{1}{4}x^2(2 \ln x - 1) + c \end{aligned}$$

$$\mathbf{b} \quad \int \frac{1}{y} dy = \int x \ln x dx$$

$$\ln|y| = \frac{1}{4}x^2(2 \ln x - 1) + c$$

$$y > 0 \quad \therefore \ln y = \frac{1}{4}x^2(2 \ln x - 1) + c$$

$$y = 4 \quad \text{when } x = 2$$

$$\therefore \ln 4 = 2 \ln 2 - 1 + c, \quad c = 1$$

$$\therefore \ln y = \frac{1}{4}x^2(2 \ln x - 1) + 1$$

$$\text{when } x = 1, \ln y = \frac{1}{4}(0 - 1) + 1 = \frac{3}{4}$$

$$\therefore y = e^{\frac{3}{4}}$$

$$2 \quad = \int_0^{\frac{\pi}{6}} \frac{1}{2} [\cos 4x + \cos(-2x)] dx$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} \cos 4x + \frac{1}{2} \cos 2x \right) dx$$

$$= \left[\frac{1}{8} \sin 4x + \frac{1}{4} \sin 2x \right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{8} \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2} \right) - (0)$$

$$= \frac{3}{16} \sqrt{3}$$

$$4 \quad \mathbf{a} \quad = \pi \int_1^4 \left(2 - \frac{1}{\sqrt{x}} \right)^2 dx$$

$$= \pi \int_1^4 (4 - 4x^{-\frac{1}{2}} + x^{-1}) dx$$

$$= \pi [4x - 8x^{\frac{1}{2}} + \ln|x|]_1^4$$

$$= \pi [(16 - 16 + \ln 4) - (4 - 8 + 0)]$$

$$= \pi(4 + \ln 4) = \pi(4 + \ln 2^2)$$

$$= \pi(4 + 2 \ln 2) = 2\pi(2 + \ln 2)$$

$$\mathbf{b} \quad = 10^3 \times 2\pi(2 + \ln 2)$$

$$= 2000\pi(2 + \ln 2) = 16\,900 \text{ cm}^3$$

$$6 \quad \mathbf{a} \quad = \int_0^{\frac{\pi}{3}} \sec x \tan x dx$$

$$= [\sec x]_0^{\frac{\pi}{3}}$$

$$= 2 - 1$$

$$= 1$$

$$\mathbf{b} \quad u = \cos \theta \quad \therefore \frac{du}{d\theta} = -\sin \theta$$

$$\theta = 0 \Rightarrow u = 1$$

$$\theta = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin \theta}{\cos^4 \theta} d\theta = \int_1^{\frac{1}{\sqrt{2}}} \frac{1}{u^4} \times (-1) du$$

$$= \int_{\frac{1}{\sqrt{2}}}^1 u^{-4} du$$

$$= \left[-\frac{1}{3} u^{-3} \right]_{\frac{1}{\sqrt{2}}}^1$$

$$= -\frac{1}{3} (1 - 2\sqrt{2})$$

$$= -\frac{1}{3} + \frac{2}{3}\sqrt{2} \quad [a = -\frac{1}{3}, b = \frac{2}{3}]$$

7 a $x = 0 \Rightarrow t = -\frac{1}{2}$

$x = 3 \Rightarrow t = 1$

b $x = 2t + 1 \therefore \frac{dx}{dt} = 2$

$\therefore \text{area} = \int_{-\frac{1}{2}}^1 \frac{1}{2-t} \times 2 dt$

$= \int_{-\frac{1}{2}}^1 \frac{2}{2-t} dt$

$= [-2 \ln |2-t|]_{-\frac{1}{2}}^1$

$= -2(0 - \ln \frac{5}{2})$

$= 2 \ln \frac{5}{2}$

c $= \pi \int_{-\frac{1}{2}}^1 \left(\frac{1}{2-t}\right)^2 \times 2 dt$

$= 2\pi \int_{-\frac{1}{2}}^1 (2-t)^{-2} dt$

$= 2\pi[(2-t)^{-1}]_{-\frac{1}{2}}^1$

$= 2\pi(1 - \frac{2}{5}) = \frac{6}{5}\pi$

9 a $A = \pi r^2 \therefore r = \sqrt{\frac{A}{\pi}}$

$P = 2\pi r = 2\pi \sqrt{\frac{A}{\pi}} = 2\sqrt{\pi A}$

$\frac{dA}{dt} = cP = c \times 2\sqrt{\pi A} = 2c\sqrt{\pi} \times \sqrt{A}$

$\therefore \frac{dA}{dt} = k\sqrt{A}$

b $\int A^{-\frac{1}{2}} dA = \int k dt$

$2A^{\frac{1}{2}} = kt + C$

$\sqrt{A} = \frac{1}{2}kt + \frac{1}{2}C = pt + q$

$\therefore A = (pt + q)^2$

c $t = 0, A = 25 \therefore \sqrt{25} = 0 + q$

$q = 5$

$t = 20, A = 40 \therefore \sqrt{40} = 20p + 5$

$p = \frac{2\sqrt{10}-5}{20}$

$A = 50 \Rightarrow 50 = \left(\frac{2\sqrt{10}-5}{20}t + 5\right)^2$

$t = (5\sqrt{2} - 5) \div \frac{2\sqrt{10}-5}{20}$

$= 31.3 \text{ minutes (3sf)}$

8 a $u = 6x, \frac{du}{dx} = 6; \frac{dv}{dx} = \cos 3x, v = \frac{1}{3} \sin 3x$

$\int 6x \cos 3x dx = 2x \sin 3x - \int 2 \sin 3x dx$

$= 2x \sin 3x + \frac{2}{3} \cos 3x + c$

b $x = 2 \sin u \therefore \frac{dx}{du} = 2 \cos u$

$x = 0 \Rightarrow u = 0$

$x = \sqrt{3} \Rightarrow u = \frac{\pi}{3}$

$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}} dx = \int_0^{\frac{\pi}{3}} \frac{1}{2 \cos u} \times 2 \cos u du$

$= \int_0^{\frac{\pi}{3}} du$

$= [u]_0^{\frac{\pi}{3}}$

$= \frac{\pi}{3} - 0$

$= \frac{\pi}{3}$

10 a $\frac{5x+1}{(1-x)(1+2x)} \equiv \frac{A}{1-x} + \frac{B}{1+2x}$

$5x + 1 \equiv A(1 + 2x) + B(1 - x)$

$x = 1 \Rightarrow 6 = 3A \Rightarrow A = 2$

$x = -\frac{1}{2} \Rightarrow -\frac{3}{2} = \frac{3}{2}B \Rightarrow B = -1$

$f(x) \equiv \frac{2}{1-x} - \frac{1}{1+2x}$

b $= \int_0^{\frac{1}{2}} \left(\frac{2}{1-x} - \frac{1}{1+2x}\right) dx$

$= [-2 \ln |1-x| - \frac{1}{2} \ln |1+2x|]_0^{\frac{1}{2}}$

$= (-2 \ln \frac{1}{2} - \frac{1}{2} \ln 2) - (0 - 0)$

$= 2 \ln 2 - \frac{1}{2} \ln 2 = \frac{3}{2} \ln 2$

c $(1-x)^{-1} =$

$1 + (-1)(-x) + \frac{(-1)(-2)}{2} (-x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2} (-x)^3 + \dots$

$= 1 + x + x^2 + x^3 + \dots$

$(1+2x)^{-1} =$

$1 + (-1)(2x) + \frac{(-1)(-2)}{2} (2x)^2 + \frac{(-1)(-2)(-3)}{3 \times 2} (2x)^3 + \dots$

$= 1 - 2x + 4x^2 - 8x^3 + \dots$

$f(x) =$

$2(1+x+x^2+x^3) - (1-2x+4x^2-8x^3) + \dots$

$f(x) = 1 + 4x - 2x^2 + 10x^3 + \dots$