

C4 INTEGRATION

Answers - Worksheet N

$$\begin{aligned}
 1 \quad &= \left[\frac{1}{4} \times 8 \ln |4x - 3| \right]_2^7 \\
 &= 2[\ln |4x - 3|]_2^7 \\
 &= 2(\ln 25 - \ln 5) \\
 &= 2 \ln \frac{25}{5} \\
 &= 2 \ln 5 \\
 &= \ln 5^2 \\
 &= \ln 25
 \end{aligned}$$

$$\begin{aligned}
 2 \quad &\frac{dy}{dx} = \frac{x}{\cos y \sin^3 y} \\
 &\int \cos y \sin^3 y \, dy = \int x \, dx \\
 &\frac{1}{4} \sin^4 y = \frac{1}{2} x^2 + c \\
 &\sin^4 y = 2x^2 + k \\
 &y = \frac{\pi}{4} \text{ when } x = 1 \\
 &\therefore \left(\frac{1}{\sqrt{2}}\right)^4 = 2 + k \\
 &\frac{1}{4} = 2 + k \\
 &k = -\frac{7}{4} \\
 &\therefore \sin^4 y = 2x^2 - \frac{7}{4}
 \end{aligned}$$

$$\begin{array}{c}
 3 \quad \mathbf{a} \quad \begin{array}{cccccc}
 x & 0 & 0.5 & 1 & 1.5 \\
 e^{x^2-1} & 0.3679 & 0.4724 & 1 & 3.4903
 \end{array} \\
 \therefore \text{integral} \approx \frac{1}{2} \times 0.5 \times [0.3679 + 3.4903 + 2(0.4724 + 1)] = 1.70 \text{ (3sf)}
 \end{array}$$

$$\begin{array}{c}
 \mathbf{b} \quad \begin{array}{ccccccccc}
 x & 0 & 0.25 & 0.5 & 0.75 & 1 & 1.25 & 1.5 \\
 e^{x^2-1} & 0.3679 & 0.3916 & 0.4724 & 0.6456 & 1 & 1.7551 & 3.4903
 \end{array} \\
 \therefore \text{integral} \approx \frac{1}{2} \times 0.25 \times [0.3679 + 3.4903 + 2(0.3916 + 0.4724 + 0.6456 + 1 + 1.7551)] \\
 = 1.55 \text{ (3sf)}
 \end{array}$$

$$\begin{aligned}
 4 \quad \mathbf{a} \quad &\frac{3(2-x)}{(1-2x)^2(1+x)} \equiv \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+x} \\
 &3(2-x) \equiv A(1-2x)(1+x) + B(1+x) + C(1-2x)^2 \\
 &x = \frac{1}{2} \Rightarrow \frac{9}{2} = \frac{3}{2}B \Rightarrow B = 3 \\
 &x = -1 \Rightarrow 9 = 9C \Rightarrow C = 1 \\
 &\text{coeffs } x^2 \Rightarrow 0 = -2A + 4C \Rightarrow A = 2 \\
 &f(x) \equiv \frac{2}{1-2x} + \frac{3}{(1-2x)^2} + \frac{1}{1+x}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad &= \int_1^2 \left(\frac{2}{1-2x} + \frac{3}{(1-2x)^2} + \frac{1}{1+x} \right) dx \\
 &= [-\ln |1-2x| + \frac{3}{2}(1-2x)^{-1} + \ln |1+x|]_1^2 \\
 &= (-\ln 3 - \frac{1}{2} + \ln 3) - (0 - \frac{3}{2} + \ln 2) \\
 &= 1 - \ln 2
 \end{aligned}$$

$$\begin{aligned}
 5 \quad \mathbf{a} \quad &\frac{dN}{dt} = kN \\
 &\int \frac{1}{N} \, dN = \int k \, dt \\
 &\ln |N| = kt + c \\
 &N = e^{kt+c} = e^c \times e^{kt} \\
 &\therefore N = Ae^{kt}
 \end{aligned}$$

$$\mathbf{b} \quad t = 0, N = 200 \quad \therefore A = 200$$

$$t = 2, N = 3000 \quad \therefore 3000 = 200e^{2k}$$

$$\therefore k = \frac{1}{2} \ln 15 = 1.354$$

$$\therefore N = 200e^{1.354t}$$

$$\therefore 10\,000 = 200e^{1.354t}$$

$$t = \frac{1}{1.354} \ln 50 = 2.889 \text{ hours}$$

$$= 2 \text{ hours } 53 \text{ minutes}$$

$$\mathbf{c} \quad 5 \text{ per second} = 18\,000 \text{ per hour}$$

$$\frac{dN}{dt} = 200 \times 0.1354e^{1.354t}$$

$$\therefore 18\,000 = 270.8e^{1.354t}$$

$$t = \frac{1}{1.354} \ln \frac{18\,000}{270.8} = 3.099 \text{ hours}$$

$$= 3 \text{ hours } 6 \text{ minutes}$$

$$\begin{aligned}
 6 \quad \mathbf{a} &= \int_0^4 (2x+1)^{-\frac{1}{2}} dx \\
 &= \left[\frac{1}{2} \times 2(2x+1)^{\frac{1}{2}} \right]_0^4 \\
 &= \left[(2x+1)^{\frac{1}{2}} \right]_0^4 \\
 &= 3 - 1 = 2
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} &= \pi \int_0^4 \left(\frac{1}{\sqrt{2x+1}} \right)^2 dx \\
 &= \pi \int_0^4 \frac{1}{2x+1} dx \\
 &= \pi \left[\frac{1}{2} \ln |2x+1| \right]_0^4 \\
 &= \frac{1}{2} \pi (\ln 9 - 0) = \pi \ln 9^{\frac{1}{2}} \\
 &= \pi \ln 3
 \end{aligned}$$

$$8 \quad \mathbf{a} \quad \sin(A+B) \equiv \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) \equiv \sin A \cos B - \cos A \sin B$$

adding

$$2 \sin A \cos B \equiv \sin(A+B) + \sin(A-B)$$

$$\mathbf{b} \quad y=0 \Rightarrow \sin 4t=0 \Rightarrow t=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$$

curve in 1st quadrant: $0 \leq t \leq \frac{\pi}{4}$

$$x = 2 \sin 2t \quad \therefore \frac{dx}{dt} = 4 \cos 2t$$

area in 1st quadrant

$$= \int_0^{\frac{\pi}{4}} \sin 4t \times 4 \cos 2t dt$$

total area = $4 \times$ area in 1st quadrant

$$= \int_0^{\frac{\pi}{4}} 16 \sin 4t \cos 2t dt$$

$$\mathbf{c} = 8 \int_0^{\frac{\pi}{4}} (\sin 6t + \sin 2t) dt$$

$$= 8 \left[-\frac{1}{6} \cos 6t - \frac{1}{2} \cos 2t \right]_0^{\frac{\pi}{4}}$$

$$= 8 \left[(0 - 0) - \left(-\frac{1}{6} - \frac{1}{2} \right) \right] = 5\frac{1}{3}$$

$$10 \quad \mathbf{a} = \frac{1}{2} \int (1 - \cos 2x) dx = \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + c = \frac{1}{4} (2x - \sin 2x) + c$$

$$\mathbf{b} \quad u = x, \frac{du}{dx} = 1; \quad \frac{dv}{dx} = \sin^2 x, v = \frac{1}{2}x - \frac{1}{4} \sin 2x$$

$$\int x \sin^2 x dx = x \left(\frac{1}{2}x - \frac{1}{4} \sin 2x \right) - \int \left(\frac{1}{2}x - \frac{1}{4} \sin 2x \right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{4}x \sin 2x - \left(\frac{1}{4}x^2 + \frac{1}{8} \cos 2x \right) + c$$

$$= \frac{1}{4}x^2 - \frac{1}{4}x \sin 2x - \frac{1}{8} \cos 2x + c$$

$$= \frac{1}{8} (2x^2 - 2x \sin 2x - \cos 2x) + c$$

$$\mathbf{c} = \pi \int_0^{\pi} (x^{\frac{1}{2}} \sin x)^2 dx = \pi \int_0^{\pi} x \sin^2 x dx$$

$$= \frac{1}{8} \pi [2x^2 - 2x \sin 2x - \cos 2x]_0^{\pi}$$

$$= \frac{1}{8} \pi [(2\pi^2 - 0 - 1) - (0 - 0 - 1)] = \frac{1}{4} \pi^3$$

$$7 \quad u^2 = x + 3 \quad \therefore x = u^2 - 3, \quad \frac{dx}{du} = 2u$$

$$x = 0 \Rightarrow u = \sqrt{3}$$

$$x = 1 \Rightarrow u = 2$$

$$\int_0^1 x \sqrt{x+3} dx = \int_{\sqrt{3}}^2 (u^2 - 3)u \times 2u du$$

$$= \int_{\sqrt{3}}^2 (2u^4 - 6u^2) du$$

$$= \left[\frac{2}{5} u^5 - 2u^3 \right]_{\sqrt{3}}^2$$

$$= \left(\frac{64}{5} - 16 \right) - \left(\frac{2}{5} \times 9\sqrt{3} - 2 \times 3\sqrt{3} \right)$$

$$= -\frac{16}{5} - \left(-\frac{12}{5}\sqrt{3} \right)$$

$$= \frac{4}{5} (3\sqrt{3} - 4) \quad [k = \frac{4}{5}]$$

$$9 \quad \mathbf{a} \quad \frac{x^2 - 22}{(x+2)(x-4)} \equiv A + \frac{B}{x+2} + \frac{C}{x-4}$$

$$x^2 - 22 \equiv A(x+2)(x-4) + B(x-4) + C(x+2)$$

$$x = -2 \Rightarrow -18 = -6B \Rightarrow B = 3$$

$$x = 4 \Rightarrow -6 = 6C \Rightarrow C = -1$$

$$\text{coeffs } x^2 \quad \Rightarrow A = 1$$

$$\mathbf{b} = \int_0^2 \left(1 + \frac{3}{x+2} - \frac{1}{x-4} \right) dx$$

$$= [x + 3 \ln |x+2| - \ln |x-4|]_0^2$$

$$= (2 + 3 \ln 4 - \ln 2) - (0 + 3 \ln 2 - \ln 4)$$

$$= 2 + 6 \ln 2 - \ln 2 - 3 \ln 2 + 2 \ln 2$$

$$= 2 + 4 \ln 2$$

$$= 2 + \ln 16$$