C4 INTEGRATION — Worksheet L

1 **a** Express $\dfrac{x+4}{(1+x)(2-x)}$ in partial fractions.

b Given that $y = 2$ when $x = 3$, solve the differential equation

$$\frac{dy}{dx} = \frac{y(x+4)}{(1+x)(2-x)}.$$

2 Given that $y = 0$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^{x+y}\cos x.$$

3 Given that $\dfrac{dy}{dx}$ is inversely proportional to x and that $y = 4$ and $\dfrac{dy}{dx} = \frac{5}{3}$ when $x = 3$, find an expression for y in terms of x.

4 A quantity has the value N at time t hours and is increasing at a rate proportional to N.

a Write down a differential equation relating N and t.

b By solving your differential equation, show that

$$N = Ae^{kt},$$

where A and k are constants and k is positive.

Given that when $t = 0$, $N = 40$ and that when $t = 5$, $N = 60$,

c find the values of A and k,

d find the value of N when $t = 12$.

5 A cube is increasing in size and has volume V cm^3 and surface area A cm^2 at time t seconds.

a Show that

$$\frac{dV}{dA} = k\sqrt{A},$$

where k is a positive constant.

Given that the rate at which the volume of the cube is increasing is proportional to its surface area and that when $t = 10$, $A = 100$ and $\dfrac{dA}{dt} = 5$,

b show that

$$A = \tfrac{1}{16}(t + 30)^2.$$

6 At time $t = 0$, a piece of radioactive material has mass 24 g. Its mass after t days is m grams and is decreasing at a rate proportional to m.

a By forming and solving a suitable differential equation, show that

$$m = 24e^{-kt},$$

where k is a positive constant.

After 20 days, the mass of the material is found to be 22.6 g.

b Find the value of k.

c Find the rate at which the mass is decreasing after 20 days.

d Find how long it takes for the mass of the material to be halved.

7 A quantity has the value P at time t seconds and is decreasing at a rate proportional to $\sqrt{P}$.

a By forming and solving a suitable differential equation, show that

$$P = (a - bt)^2,$$

where a and b are constants.

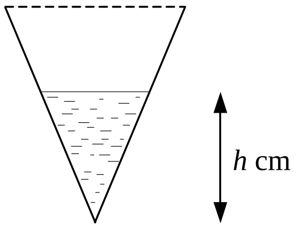
Given that when $t = 0$, $P = 400$,

b find the value of a.

Given also that when $t = 30$, $P = 100$,

c find the value of P when $t = 50$.

8



The diagram shows a container in the shape of a right-circular cone. A quantity of water is poured into the container but this then leaks out from a small hole at its vertex.

In a model of the situation it is assumed that the rate at which the volume of water in the container, V cm^3, decreases is proportional to V. Given that the depth of the water is h cm at time t minutes,

a show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -kh,$$

where k is a positive constant.

Given also that $h = 12$ when $t = 0$ and that $h = 10$ when $t = 20$,

b show that

$$h = 12\mathrm{e}^{-kt},$$

and find the value of k,

c find the value of t when $h = 6$.

9 **a** Express $\dfrac{1}{(1+x)(1-x)}$ in partial fractions.

In an industrial process, the mass of a chemical, m kg, produced after t hours is modelled by the differential equation

$$\frac{\mathrm{d}m}{\mathrm{d}t} = k\mathrm{e}^{-t}(1 + m)(1 - m),$$

where k is a positive constant.

Given that when $t = 0$, $m = 0$ and that the initial rate at which the chemical is produced is 0.5 kg per hour,

b find the value of k,

c show that, for $0 \le m < 1$, $\ln\left(\dfrac{1+m}{1-m}\right) = 1 - \mathrm{e}^{-t}$.

d find the time taken to produce 0.1 kg of the chemical,

e show that however long the process is allowed to run, the maximum amount of the chemical that will be produced is about 462 g.