⟨**C4**⟩ **INTEGRATION**   *Answers - Worksheet F*

1 $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = \cos x,\ v = \sin x$

$\displaystyle \int x \cos x\ dx = x \sin x - \int \sin x\ dx$

$= x \sin x + \cos x + c$

2   **a** $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = e^x,\ v = e^x$ **b** $u = 4x,\ \dfrac{du}{dx} = 4;\ \dfrac{dv}{dx} = \sin x,\ v = -\cos x$

$\displaystyle \int xe^x\ dx = xe^x - \int e^x\ dx$ $\displaystyle \int 4x \sin x\ dx = -4x \cos x - \int -4 \cos x\ dx$

$= xe^x - e^x + c$ $= -4x \cos x + \displaystyle\int 4 \cos x\ dx$

$= e^x(x - 1) + c$ $= -4x \cos x + 4 \sin x + c$

c $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = \cos 2x,\ v = \tfrac{1}{2}\sin 2x$ **d** $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = (x+1)^{\frac{1}{2}},\ v = \tfrac{2}{3}(x+1)^{\frac{3}{2}}$

$\displaystyle \int x \cos 2x\ dx = \tfrac{1}{2}x \sin 2x - \int \tfrac{1}{2}\sin 2x\ dx$ $\displaystyle \int x\sqrt{x+1}\ dx = \tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \int \tfrac{2}{3}(x+1)^{\frac{3}{2}}\ dx$

$= \tfrac{1}{2}x \sin 2x + \tfrac{1}{4}\cos 2x + c$ $= \tfrac{2}{3}x(x+1)^{\frac{3}{2}} - \tfrac{4}{15}(x+1)^{\frac{5}{2}} + c$

$= \tfrac{2}{15}(x+1)^{\frac{3}{2}}[5x - 2(x+1)] + c$

$= \tfrac{2}{15}(3x - 2)(x+1)^{\frac{3}{2}} + c$

e $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = e^{-3x},\ v = -\tfrac{1}{3}e^{-3x}$ **f** $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = \sec^2 x,\ v = \tan x$

$\displaystyle \int \frac{x}{e^{3x}}\ dx = -\tfrac{1}{3}xe^{-3x} - \int -\tfrac{1}{3}e^{-3x}\ dx$ $\displaystyle \int x \sec^2 x\ dx = x \tan x - \int \tan x\ dx$

$= -\tfrac{1}{3}xe^{-3x} + \displaystyle\int \tfrac{1}{3}e^{-3x}\ dx$ $= x \tan x + \displaystyle\int \frac{-\sin x}{\cos x}\ dx$

$= -\tfrac{1}{3}xe^{-3x} - \tfrac{1}{9}e^{-3x} + c$ $= x \tan x + \ln|\cos x| + c$

$= -\tfrac{1}{9}e^{-3x}(3x + 1) + c$

3   **i** $u = x,\ \dfrac{du}{dx} = 1;\ \dfrac{dv}{dx} = (2x+1)^3,\ v = \tfrac{1}{8}(2x+1)^4$

$\displaystyle \int x(2x+1)^3\ dx = \tfrac{1}{8}x(2x+1)^4 - \int \tfrac{1}{8}(2x+1)^4\ dx$

$= \tfrac{1}{8}x(2x+1)^4 - \tfrac{1}{80}(2x+1)^5 + c$

$= \tfrac{1}{80}(2x+1)^4[10x - (2x+1)] + c$

$= \tfrac{1}{80}(8x - 1)(2x+1)^4 + c$

ii $u = 2x + 1\ \therefore\ x = \tfrac{1}{2}(u-1),\ \dfrac{du}{dx} = 2$

$\displaystyle \int x(2x+1)^3\ dx = \int \tfrac{1}{2}(u-1)u^3 \times \tfrac{1}{2}\ du$

$= \tfrac{1}{4}\displaystyle\int (u^4 - u^3)\ du$

$= \tfrac{1}{4}(\tfrac{1}{5}u^5 - \tfrac{1}{4}u^4) + c$

$= \tfrac{1}{4}[\tfrac{1}{5}(2x+1)^5 - \tfrac{1}{4}(2x+1)^4] + c$

$= \tfrac{1}{80}(2x+1)^4[4(2x+1) - 5] + c$

$= \tfrac{1}{80}(8x - 1)(2x+1)^4 + c,\ $ as for part **i**

4 $u = x, \dfrac{du}{dx} = 1; \quad \dfrac{dv}{dx} = e^{-x}, v = -e^{-x}$

$\displaystyle\int_0^2 x e^{-x} \, dx = [-xe^{-x}]_0^2 - \int_0^2 -e^{-x} \, dx = [-xe^{-x}]_0^2 + \int_0^2 e^{-x} \, dx$

$\qquad\qquad\qquad = [-xe^{-x} - e^{-x}]_0^2 = (-2e^{-2} - e^{-2}) - (0 - 1)$

$\qquad\qquad\qquad = 1 - 3e^{-2}$

5 **a** $u = x, \dfrac{du}{dx} = 1; \quad \dfrac{dv}{dx} = \cos x, v = \sin x$ **b** $u = x, \dfrac{du}{dx} = 1; \quad \dfrac{dv}{dx} = e^{2x}, v = \tfrac{1}{2}e^{2x}$

$\displaystyle\int_0^{\frac{\pi}{6}} x \cos x \, dx = [x \sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x \, dx$ $\qquad$ $\displaystyle\int_0^1 x e^{2x} \, dx = [\tfrac{1}{2} x e^{2x}]_0^1 - \int_0^1 \tfrac{1}{2} e^{2x} \, dx$

$\qquad\qquad\qquad = [x \sin x + \cos x]_0^{\frac{\pi}{6}}$ $\qquad\qquad\qquad\qquad = [\tfrac{1}{2} x e^{2x} - \tfrac{1}{4} e^{2x}]_0^1$

$\qquad\qquad\qquad = (\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{2}) - (0 + 1)$ $\qquad\qquad\qquad\qquad = (\tfrac{1}{2} e^2 - \tfrac{1}{4} e^2) - (0 - \tfrac{1}{4})$

$\qquad\qquad\qquad = \tfrac{1}{12}(\pi + 6\sqrt{3} - 12)$ $\qquad\qquad\qquad\qquad = \tfrac{1}{4}(e^2 + 1)$

c $u = x, \dfrac{du}{dx} = 1; \quad \dfrac{dv}{dx} = \sin 3x, v = -\tfrac{1}{3}\cos 3x$

$\displaystyle\int_0^{\frac{\pi}{4}} x \sin 3x \, dx = [-\tfrac{1}{3} x \cos 3x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\tfrac{1}{3}\cos 3x \, dx$

$\qquad\qquad\qquad = [-\tfrac{1}{3} x \cos 3x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \tfrac{1}{3}\cos 3x \, dx$

$\qquad\qquad\qquad = [-\tfrac{1}{3} x \cos 3x + \tfrac{1}{9}\sin 3x]_0^{\frac{\pi}{4}}$

$\qquad\qquad\qquad = [-\tfrac{\pi}{12}(-\tfrac{1}{\sqrt{2}}) + \tfrac{1}{9}(\tfrac{1}{\sqrt{2}})] - (0)$

$\qquad\qquad\qquad = \tfrac{1}{72}\sqrt{2}\,(3\pi + 4)$

6 **a** $u = x^2, \dfrac{du}{dx} = 2x; \quad \dfrac{dv}{dx} = e^x, v = e^x$

$\displaystyle\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$

for $\displaystyle\int 2x e^x \, dx, \quad u = 2x, \dfrac{du}{dx} = 2; \quad \dfrac{dv}{dx} = e^x, v = e^x$

$\qquad\qquad\qquad \displaystyle\int 2x e^x \, dx = 2x e^x - \int 2e^x \, dx$

$\qquad\qquad\qquad\qquad = 2x e^x - 2e^x + c$

$\therefore \displaystyle\int x^2 e^x \, dx = x^2 e^x - (2x e^x - 2e^x) + c$

$\qquad\qquad\qquad = e^x(x^2 - 2x + 2) + c$

b $u = e^x, \dfrac{du}{dx} = e^x; \quad \dfrac{dv}{dx} = \sin x, v = -\cos x$

$\displaystyle\int e^x \sin x \, dx = -e^x \cos x - \int -e^x \cos x \, dx$

$\qquad\qquad\qquad = -e^x \cos x + \int e^x \cos x \, dx$

for $\displaystyle\int e^x \cos x \, dx, \quad u = e^x, \dfrac{du}{dx} = e^x; \quad \dfrac{dv}{dx} = \cos x, v = \sin x$

$\qquad\qquad\qquad \displaystyle\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$\therefore \displaystyle\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$

$\quad 2\displaystyle\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + c$

$\quad \displaystyle\int e^x \sin x \, dx = \tfrac{1}{2} e^x(\sin x - \cos x) + c$

7 **a** $u = x^2,\ \dfrac{du}{dx} = 2x;\ \ \dfrac{dv}{dx} = \sin x,\ v = -\cos x$

$$\int x^2 \sin x\ dx = -x^2 \cos x - \int -2x \cos x\ dx$$

$$= -x^2 \cos x + \int 2x \cos x\ dx$$

for $\int 2x \cos x\ dx,\quad u = 2x,\ \dfrac{du}{dx} = 2;\ \ \dfrac{dv}{dx} = \cos x,\ v = \sin x$

$$\int 2x \cos x\ dx = 2x \sin x - \int 2 \sin x\ dx$$

$$= 2x \sin x + 2 \cos x + c$$

$\therefore\ \int x^2 \sin x\ dx = -x^2 \cos x + (2x \sin x + 2 \cos x) + c$

$$= (2 - x^2)\cos x + 2x \sin x + c$$

b $u = x^2,\ \dfrac{du}{dx} = 2x;\ \ \dfrac{dv}{dx} = e^{3x},\ v = \tfrac{1}{3} e^{3x}$

$$\int x^2 e^{3x}\ dx = \tfrac{1}{3} x^2 e^{3x} - \int \tfrac{2}{3} x e^{3x}\ dx$$

for $\int \tfrac{2}{3} x e^{3x}\ dx,\quad u = \tfrac{2}{3}x,\ \dfrac{du}{dx} = \tfrac{2}{3};\ \ \dfrac{dv}{dx} = e^{3x},\ v = \tfrac{1}{3} e^{3x}$

$$\int \tfrac{2}{3} x e^{3x}\ dx = \tfrac{2}{9} x e^{3x} - \int \tfrac{2}{9} e^{3x}\ dx$$

$$= \tfrac{2}{9} x e^{3x} - \tfrac{2}{27} e^{3x} + c$$

$\therefore\ \int x^2 e^{3x}\ dx = \tfrac{1}{3} x^2 e^{3x} - (\tfrac{2}{9} x e^{3x} - \tfrac{2}{27} e^{3x}) + c$

$$= \tfrac{1}{27} e^{3x}(9x^2 - 6x + 2) + c$$

c $u = e^{-x},\ \dfrac{du}{dx} = -e^{-x};\ \ \dfrac{dv}{dx} = \cos 2x,\ v = \tfrac{1}{2} \sin 2x$

$$\int e^{-x} \cos 2x\ dx = \tfrac{1}{2} e^{-x} \sin 2x - \int -\tfrac{1}{2} e^{-x} \sin 2x\ dx$$

$$= \tfrac{1}{2} e^{-x} \sin 2x + \int \tfrac{1}{2} e^{-x} \sin 2x\ dx$$

for $\int \tfrac{1}{2} e^{-x} \sin 2x\ dx,\quad u = \tfrac{1}{2} e^{-x},\ \dfrac{du}{dx} = -\tfrac{1}{2} e^{-x};\ \ \dfrac{dv}{dx} = \sin 2x,\ v = -\tfrac{1}{2} \cos 2x$

$$\int \tfrac{1}{2} e^{-x} \sin 2x\ dx = -\tfrac{1}{4} e^{-x} \cos 2x - \int \tfrac{1}{4} e^{-x} \cos 2x\ dx$$

$\therefore\ \int e^{-x} \cos 2x\ dx = \tfrac{1}{2} e^{-x} \sin 2x - \tfrac{1}{4} e^{-x} \cos 2x - \tfrac{1}{4} \int e^{-x} \cos 2x\ dx$

$\tfrac{5}{4} \int e^{-x} \cos 2x\ dx = \tfrac{1}{2} e^{-x} \sin 2x - \tfrac{1}{4} e^{-x} \cos 2x + c$

$\int e^{-x} \cos 2x\ dx = \tfrac{1}{5} e^{-x}(2 \sin 2x - \cos 2x) + c$

8 **a** $\dfrac{1}{x}$

b $u = \ln x,\ \dfrac{du}{dx} = \dfrac{1}{x};\ \ \dfrac{dv}{dx} = 1,\ v = x$

$$\int \ln x\ dx = x \ln x - \int \dfrac{1}{x} \times x\ dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + c$$

$$= x(\ln x - 1) + c$$

9   **a** $u = \ln 2x, \ \dfrac{du}{dx} = \dfrac{1}{x}; \ \dfrac{dv}{dx} = 1, \ v = x$

$\displaystyle \int \ln 2x \ dx = x \ln 2x - \int \dfrac{1}{x} \times x \ dx$

$\qquad\qquad\quad = x \ln 2x - \int \ dx$

$\qquad\qquad\quad = x \ln 2x - x + c$

$\qquad\qquad\quad = x(\ln 2x - 1) + c$

b $u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}; \ \dfrac{dv}{dx} = 3x, \ v = \dfrac{3}{2}x^2$

$\displaystyle \int 3x \ln x \ dx = \dfrac{3}{2}x^2 \ln x - \int \dfrac{1}{x} \times \dfrac{3}{2}x^2 \ dx$

$\qquad\qquad\quad = \dfrac{3}{2}x^2 \ln x - \int \dfrac{3}{2}x \ dx$

$\qquad\qquad\quad = \dfrac{3}{2}x^2 \ln x - \dfrac{3}{4}x^2 + c$

$\qquad\qquad\quad = \dfrac{3}{4}x^2(2 \ln x - 1) + c$

c $u = (\ln x)^2, \ \dfrac{du}{dx} = 2(\ln x) \times \dfrac{1}{x}; \ \dfrac{dv}{dx} = 1, \ v = x$

$\displaystyle \int (\ln x)^2 \ dx = x(\ln x)^2 - \int 2 \ln x \ dx$

for $\displaystyle \int 2 \ln x \ dx, \quad u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}; \ \dfrac{dv}{dx} = 2, \ v = 2x$

$\qquad\qquad \displaystyle \int 2 \ln x \ dx = 2x \ln x - \int 2 \ dx$

$\qquad\qquad\qquad\qquad = 2x \ln x - 2x + c$

$\therefore \ \displaystyle \int (\ln x)^2 \ dx = x(\ln x)^2 - (2x \ln x - 2x) + c$

$\qquad\qquad\qquad = x[(\ln x)^2 - 2 \ln x + 2] + c$

10 **a** $u = x + 2, \ \dfrac{du}{dx} = 1; \ \dfrac{dv}{dx} = e^x, \ v = e^x$

$\displaystyle \int_{-1}^{0} (x + 2)e^x \ dx$

$\qquad = \left[(x + 2)e^x \right]_{-1}^{0} - \int_{-1}^{0} e^x \ dx$

$\qquad = \left[(x + 2)e^x - e^x \right]_{-1}^{0}$

$\qquad = (2 - 1) - (e^{-1} - e^{-1})$

$\qquad = 1$

b $u = \ln x, \ \dfrac{du}{dx} = \dfrac{1}{x}; \ \dfrac{dv}{dx} = x^2, \ v = \dfrac{1}{3}x^3$

$\displaystyle \int_{1}^{2} x^2 \ln x \ dx$

$\qquad = \left[\dfrac{1}{3}x^3 \ln x \right]_{1}^{2} - \int_{1}^{2} \dfrac{1}{3}x^2 \ dx$

$\qquad = \left[\dfrac{1}{3}x^3 \ln x - \dfrac{1}{9}x^3 \right]_{1}^{2}$

$\qquad = \left(\dfrac{8}{3} \ln 2 - \dfrac{8}{9} \right) - \left(0 - \dfrac{1}{9} \right)$

$\qquad = \dfrac{8}{3} \ln 2 - \dfrac{7}{9}$

c $u = 2x, \ \dfrac{du}{dx} = 2; \ \dfrac{dv}{dx} = e^{3x - 1}, \ v = \dfrac{1}{3}e^{3x - 1}$

$\displaystyle \int_{\frac{1}{3}}^{1} 2xe^{3x - 1} \ dx$

$\qquad = \left[\dfrac{2}{3}xe^{3x - 1} \right]_{\frac{1}{3}}^{1} - \int_{\frac{1}{3}}^{1} \dfrac{2}{3}e^{3x - 1} \ dx$

$\qquad = \left[\dfrac{2}{3}xe^{3x - 1} - \dfrac{2}{9}e^{3x - 1} \right]_{\frac{1}{3}}^{1}$

$\qquad = \left(\dfrac{2}{3}e^2 - \dfrac{2}{9}e^2 \right) - \left(\dfrac{2}{9} - \dfrac{2}{9} \right)$

$\qquad = \dfrac{4}{9}e^2$

d $u = \ln (2x + 3), \ \dfrac{du}{dx} = \dfrac{2}{2x + 3}; \ \dfrac{dv}{dx} = 1, \ v = x$

$\displaystyle \int_{0}^{3} \ln (2x + 3) \ dx$

$\qquad = \left[x \ln (2x + 3) \right]_{0}^{3} - \int_{0}^{3} \dfrac{2x}{2x + 3} \ dx$

$\qquad = \left[x \ln (2x + 3) \right]_{0}^{3} - \int_{0}^{3} \dfrac{(2x + 3) - 3}{2x + 3} \ dx$

$\qquad = \left[x \ln (2x + 3) \right]_{0}^{3} - \int_{0}^{3} \left(1 - \dfrac{3}{2x + 3} \right) \ dx$

$\qquad = \left[x \ln (2x + 3) - x + \dfrac{3}{2} \ln \left| 2x + 3 \right| \right]_{0}^{3}$

$\qquad = \left(3 \ln 9 - 3 + \dfrac{3}{2} \ln 9 \right) - \left(0 - 0 + \dfrac{3}{2} \ln 3 \right)$

$\qquad = \dfrac{15}{2} \ln 3 - 3$

e $u = x^2, \frac{du}{dx} = 2x; \frac{dv}{dx} = \cos x, v = \sin x$

$\int x^2 \cos x \ dx = x^2 \sin x - \int 2x \sin x \ dx$

for $\int 2x \sin x \ dx$, $u = 2x, \frac{du}{dx} = 2; \frac{dv}{dx} = \sin x, v = -\cos x$

$\int 2x \sin x \ dx = -2x \cos x - \int -2 \cos x \ dx$

$= -2x \cos x + \int 2 \cos x \ dx$

$= -2x \cos x + 2 \sin x + c$

$\therefore \int_0^{\frac{\pi}{2}} x^2 \cos x \ dx = [x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\frac{\pi}{2}}$

$= (\frac{1}{4}\pi^2 + 0 - 2) - (0 + 0 - 0)$

$= \frac{1}{4}\pi^2 - 2$

f $u = e^{3x}, \frac{du}{dx} = 3e^{3x}; \frac{dv}{dx} = \sin 2x, v = -\frac{1}{2}\cos 2x$

$\int e^{3x} \sin 2x \ dx = -\frac{1}{2}e^{3x} \cos 2x - \int -\frac{3}{2}e^{3x} \cos 2x \ dx$

$= -\frac{1}{2}e^{3x} \cos 2x + \int \frac{3}{2}e^{3x} \cos 2x \ dx$

for $\int \frac{3}{2}e^{3x} \cos 2x \ dx$, $u = \frac{3}{2}e^{3x}, \frac{du}{dx} = \frac{9}{2}e^{3x}; \frac{dv}{dx} = \cos 2x, v = \frac{1}{2}\sin 2x$

$\int \frac{3}{2}e^{3x} \cos 2x \ dx = \frac{3}{4}e^{3x} \sin 2x - \int \frac{9}{4}e^{3x} \sin 2x \ dx$

$\therefore \int e^{3x} \sin 2x \ dx = -\frac{1}{2}e^{3x} \cos 2x + \frac{3}{4}e^{3x} \sin 2x - \int \frac{9}{4}e^{3x} \sin 2x \ dx$

$\frac{13}{4} \int e^{3x} \sin 2x \ dx = -\frac{1}{2}e^{3x} \cos 2x + \frac{3}{4}e^{3x} \sin 2x + c$

$\therefore \int_0^{\frac{\pi}{4}} e^{3x} \sin 2x \ dx = \frac{4}{13} [-\frac{1}{2}e^{3x} \cos 2x + \frac{3}{4}e^{3x} \sin 2x]_0^{\frac{\pi}{4}}$

$= \frac{4}{13} [(0 + \frac{3}{4}e^{\frac{3\pi}{4}}) - (-\frac{1}{2} + 0)]$

$= \frac{1}{13} (3e^{\frac{3\pi}{4}} + 2)$