C4 DIFFERENTIATION — Answers - Worksheet F

1 **a** $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = -2t^{-2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-2t^{-2}}{2t} = -\frac{1}{t^3}$

b $t = 2 \therefore x = 4,\ y = 1$

grad $= -\frac{1}{8}$

$\therefore$ grad of normal = 8

$\therefore y - 1 = 8(x - 4)$

$y = 8x - 31$

2 $x = 1 \therefore y = 4$

$\frac{dy}{dx} = 4^x \ln 4$

grad $= 4 \ln 4 = 4 \ln 2^2 = 8 \ln 2$

$\therefore y - 4 = (8 \ln 2)(x - 1)$

$y = 4 + 8(x - 1) \ln 2$

3 **a** $\frac{dx}{d\theta} = \sec\theta\tan\theta$, $\frac{dy}{d\theta} = -2\sin 2\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-2\sin 2\theta}{\sec\theta\tan\theta}$

$= -4\sin\theta\cos\theta \times \cos\theta \times \frac{\cos\theta}{\sin\theta}$

$= -4\cos^3\theta$

b $\theta = \frac{\pi}{6} \therefore x = \frac{2}{\sqrt{3}},\ y = \frac{1}{2}$

grad $= -4 \times (\frac{\sqrt{3}}{2})^3 = -\frac{3}{2}\sqrt{3}$

$\therefore y - \frac{1}{2} = -\frac{3}{2}\sqrt{3}\,(x - \frac{2}{\sqrt{3}})$

$2y - 1 = -3\sqrt{3}\,x + 6$

$3\sqrt{3}\,x + 2y = 7 \qquad [k = 7]$

4 **a** $4x + 6 \times y + 6x \times \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$

$4x + 6y = \frac{dy}{dx}(2y - 6x)$

$\frac{dy}{dx} = \frac{2x + 3y}{y - 3x}$

at P, grad = 1

$\therefore$ grad of normal = –1

$\therefore y + 5 = -(x - 2)$

$x + y + 3 = 0$

b sub. $y = -x - 3$ into eqn of curve

$2x^2 + 6x(-x - 3) - (-x - 3)^2 + 77 = 0$

$5x^2 + 24x - 68 = 0$

$(5x + 34)(x - 2) = 0$

$x = 2$ (at P) or $-\frac{34}{5} \quad \therefore x = -6\frac{4}{5}$

5 **a** $y = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore (\frac{\pi}{2} - 1, 0), (\frac{3\pi}{2} + 1, 0)$

b $\frac{dx}{d\theta} = 1 - \cos\theta$, $\frac{dy}{d\theta} = -\sin\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{-\sin\theta}{1 - \cos\theta}$

$= -\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 - (1 - 2\sin^2\frac{\theta}{2})}$

$= -\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

c $-\cot\frac{\theta}{2} = 0$

$\frac{\theta}{2} = \frac{\pi}{2},\ \theta = \pi$

$\therefore (\pi, -1)$

6 **a** $\sin\theta = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

b $\theta = \frac{\pi}{6} \therefore y = \frac{4}{3}$

$\frac{dx}{d\theta} = \cos\theta$, $\frac{dy}{d\theta} = 2\sec\theta \times \sec\theta\tan\theta$,

$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{2\sec^2\theta\tan\theta}{\cos\theta} = \frac{2\sin\theta}{\cos^4\theta}$

$\therefore$ grad $= \frac{16}{9}$

$\therefore y - \frac{4}{3} = \frac{16}{9}(x - \frac{1}{2})$

$9y - 12 = 16x - 8$

$16x - 9y + 4 = 0$

c $y = \sec^2\theta = \frac{1}{\cos^2\theta} = \frac{1}{1 - \sin^2\theta}$

$\therefore y = \frac{1}{1 - x^2}$

7 **a** $2\cos x - 2\frac{dy}{dx}\sec^2 2y = 0$

$2\cos x = \frac{2}{\cos^2 2y}\frac{dy}{dx}$

$\frac{dy}{dx} = \cos x \cos^2 2y$

b grad = $\frac{1}{2} \times (\frac{1}{2})^2 = \frac{1}{8}$

$\therefore\ y - \frac{\pi}{6} = \frac{1}{8}(x - \frac{\pi}{3})$

$24y - 4\pi = 3x - \pi$

$3x - 24y + 3\pi = 0$

$x - 8y + \pi = 0$

8 **a** $\frac{dx}{dt} = -4\sin t,\ \frac{dy}{dt} = 3\cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3\cos t}{-4\sin t} = -\frac{3}{4}\cot t$

b $y - 3\sin t = -\frac{3\cos t}{4\sin t}(x - 4\cos t)$

$4y\sin t - 12\sin^2 t = -3x\cos t + 12\cos^2 t$

$3x\cos t + 4y\sin t = 12(\sin^2 t + \cos^2 t)$

$3x\cos t + 4y\sin t = 12$

c $\cos t = \frac{x}{4},\ \ \sin t = \frac{y}{3}$

$\cos^2 t + \sin^2 t = 1$

$\therefore\ (\frac{x}{4})^2 + (\frac{y}{3})^2 = 1$

$9x^2 + 16y^2 = 144$

9 **a** $\frac{dx}{dt} = \frac{1\times(t+1) - t\times 1}{(t+1)^2} = \frac{1}{(t+1)^2}$

$\frac{dy}{dt} = \frac{2\times(t-1) - 2t\times 1}{(t-1)^2} = \frac{-2}{(t-1)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-2}{(t-1)^2} \div \frac{1}{(t+1)^2}$

$= \frac{-2(t+1)^2}{(t-1)^2} = -2\left(\frac{t+1}{t-1}\right)^2$

b at O, $t = 0$ $\therefore$ grad $= -2$

$\therefore$ grad of normal $= \frac{1}{2}$

$\therefore\ y = \frac{1}{2}x$

c $\frac{2t}{t-1} = \frac{1}{2} \times \frac{t}{t+1}$

$4t(t + 1) = t(t - 1)$

$3t^2 + 5t = 0$

$t(3t + 5) = 0$

$t = 0$ (at O) or $-\frac{5}{3}$

$\therefore\ (\frac{5}{2}, \frac{5}{4})$

d $x = \frac{t}{t+1} \Rightarrow xt + x = t$

$x = t(1 - x)$

$t = \frac{x}{1-x}$

$\therefore\ y = \frac{\frac{2x}{1-x}}{\frac{x}{1-x} - 1}$

$y = \frac{2x}{x - (1-x)}$

$y = \frac{2x}{2x - 1}$