1 Given that $\operatorname{cosec}^2\theta - \cot\theta = 3$, show that $\cot^2\theta - \cot\theta - 2 = 0$.

Hence solve the equation $\operatorname{cosec}^2\theta - \cot\theta = 3$ for $0° \leqslant \theta \leqslant 180°$. **[6]**

2 Archimedes, about 2200 years ago, used regular polygons inside and outside circles to obtain approximations for π.

(i) Fig. 8.1 shows a regular 12-sided polygon inscribed in a circle of radius 1 unit, centre O. AB is one of the sides of the polygon. C is the midpoint of AB. Archimedes used the fact that the circumference of the circle is greater than the perimeter of this polygon.

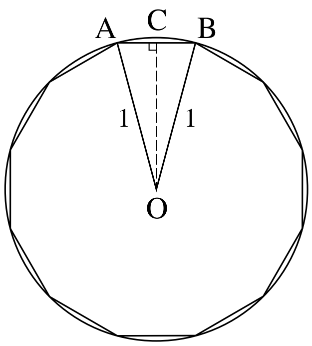

Fig. 8.1

(A) Show that AB = $2\sin 15°$. **[2]**

(B) Use a double angle formula to express $\cos 30°$ in terms of $\sin 15°$. Using the exact value of $\cos 30°$, show that $\sin 15° = \frac{1}{2}\sqrt{2-\sqrt{3}}$. **[4]**

(C) Use this result to find an exact expression for the perimeter of the polygon.

Hence show that $\pi > 6\sqrt{2-\sqrt{3}}$. **[2]**

(ii) In Fig. 8.2, a regular 12-sided polygon lies outside the circle of radius 1 unit, which touches each side of the polygon. F is the midpoint of DE. Archimedes used the fact that the circumference of the circle is less than the perimeter of this polygon.

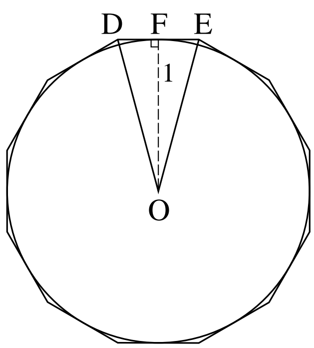

Fig. 8.2

(A) Show that DE = $2\tan 15°$. **[2]**

(B) Let $t = \tan 15°$. Use a double angle formula to express $\tan 30°$ in terms of t.

Hence show that $t^2 + 2\sqrt{3}t - 1 = 0$. **[3]**

(C) Solve this equation, and hence show that $\pi < 12\left(2-\sqrt{3}\right)$. **[4]**

(iii) Use the results in parts **(i)**(C) and **(ii)**(C) to establish upper and lower bounds for the value of π, giving your answers in decimal form. **[2]**

3 Express $\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants to be determined, and $0° < \alpha < 90°$.

Hence solve the equation $\sin\theta - 3\cos\theta = 1$ for $0° \leqslant \theta \leqslant 360°$. [7]

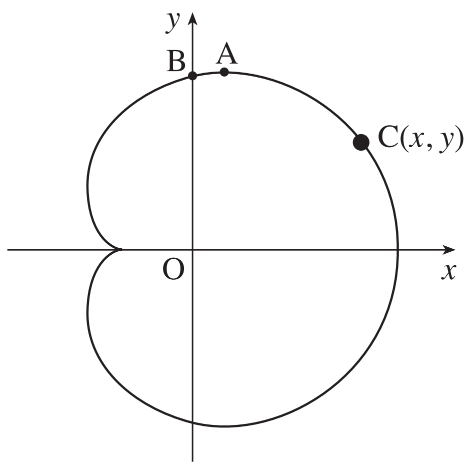

Fig. 8

In a theme park ride, a capsule C moves in a vertical plane (see Fig. 8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta, \qquad (0 \leqslant \theta < 2\pi),$$

where x and y are in metres.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$.

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \frac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C. [6]

(ii) Express $x^2 + y^2$ in terms of θ. Hence show that

$$x^2 + y^2 = 125 + 100\cos\theta. \quad [4]$$

(iii) Using this result, or otherwise, find the greatest and least distances of C from O. [2]

You are given that, at the point B on the path vertically above O,

$$2\cos^2\theta + 2\cos\theta - 1 = 0.$$

(iv) Using this result, and the result in part (**ii**), find the distance OB. Give your answer to 3 significant figures. [4]

5 Show that $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta}$.

Hence solve the equation

$$\cot 2\theta = 1 + \tan\theta \quad \text{for } 0^\circ < \theta < 360^\circ.$$ **[7]**