1 Fig. 8 illustrates a hot air balloon on its side. The balloon is modelled by the volume of revolution about the x-axis of the curve with parametric equations

$$x = 2 + 2\sin\theta, \quad y = 2\cos\theta + \sin 2\theta, \quad (0 \leqslant \theta \leqslant 2\pi).$$

The curve crosses the x-axis at the point A (4, 0). B and C are maximum and minimum points on the curve. Units on the axes are metres.

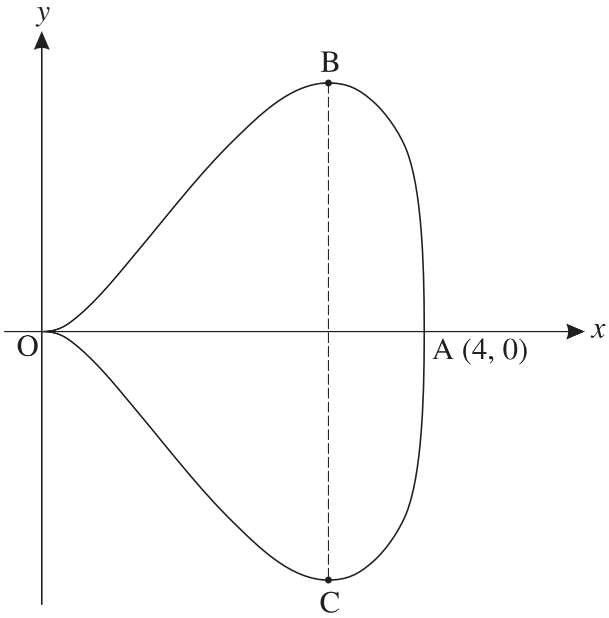

Fig. 8

(i) Find $\frac{dy}{dx}$ in terms of θ. **[4]**

(ii) Verify that $\frac{dy}{dx} = 0$ when $\theta = \frac{1}{6}\pi$, and find the exact coordinates of B.

Hence find the maximum width BC of the balloon. **[5]**

(iii) (A) Show that $y = x\cos\theta$.

(B) Find $\sin\theta$ in terms of x and show that $\cos^2\theta = x - \frac{1}{4}x^2$.

(C) Hence show that the cartesian equation of the curve is $y^2 = x^3 - \frac{1}{4}x^4$. **[7]**

(iv) Find the volume of the balloon. **[3]**

2 A curve has equation

$$x^2 + 4y^2 = k^2,$$

where k is a positive constant.

(i) Verify that

$$x = k\cos\theta, \qquad y = \tfrac{1}{2}k\sin\theta,$$

are parametric equations for the curve. [3]

(ii) Hence or otherwise show that $\dfrac{dy}{dx} = -\dfrac{x}{4y}$. [3]

(iii) Fig. 8 illustrates the curve for a particular value of k. Write down this value of k. [1]

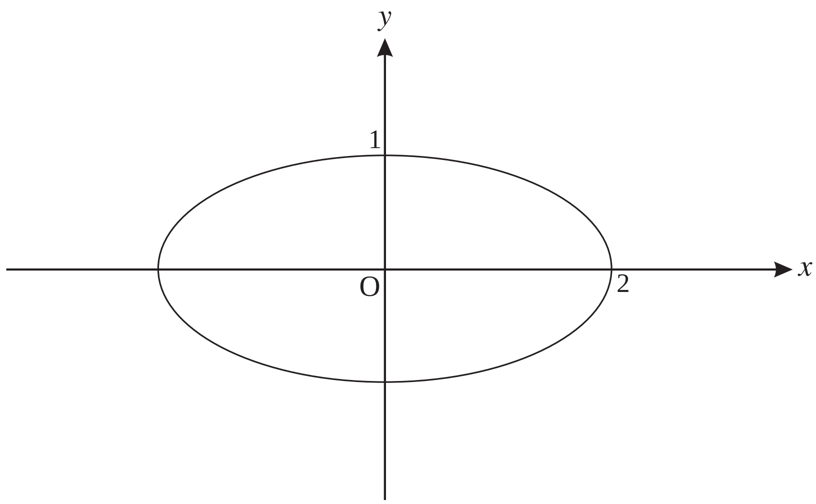

Fig. 8

(iv) Copy Fig. 8 and on the same axes sketch the curves for $k = 1$, $k = 3$ and $k = 4$. [3]

On a map, the curves represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(v) Explain why the path of the stream is modelled by the differential equation

$$\frac{dy}{dx} = \frac{4y}{x}.$$ [2]

(vi) Solve this differential equation.

Given that the path of the stream passes through the point $(2, 1)$, show that its equation is $y = \dfrac{x^4}{16}$. [6]

3 A curve is defined parametrically by the equations

$$x = t - \ln t, \qquad y = t + \ln t \qquad (t > 0).$$

Find the gradient of the curve at the point where $t = 2$. [5]

4 Fig. 7a shows the curve with the parametric equations

$$x = 2\cos\theta, \quad y = \sin 2\theta, \quad -\tfrac{\pi}{2} \leqslant \theta \leqslant \tfrac{\pi}{2}.$$

The curve meets the x-axis at O and P. Q and R are turning points on the curve. The scales on the axes are the same.

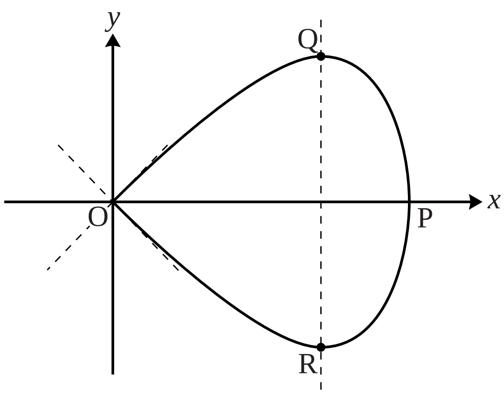

Fig. 7a

(i) State, with their coordinates, the points on the curve for which $\theta = -\frac{\pi}{2}$, $\theta = 0$ and $\theta = \frac{\pi}{2}$. **[3]**

(ii) Find $\dfrac{dy}{dx}$ in terms of θ. Hence find the gradient of the curve when $\theta = \frac{\pi}{2}$, and verify that the two tangents to the curve at the origin meet at right angles. **[5]**

(iii) Find the exact coordinates of the turning point Q. **[3]**

When the curve is rotated about the x-axis, it forms a paperweight shape, as shown in Fig. 7b.

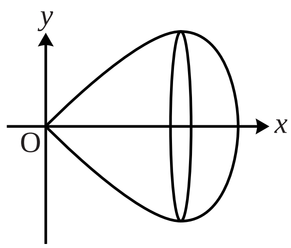

Fig. 7b

(iv) Express $\sin^2\theta$ in terms of x. Hence show that the cartesian equation of the curve is $y^2 = x^2(1 - \frac{1}{4}x^2)$. **[4]**

(v) Find the volume of the paperweight shape. **[4]**

5 **(i)** Express $\dfrac{3}{(y-2)(y+1)}$ in partial fractions. **[3]**

(ii) Hence, given that x and y satisfy the differential equation

$$\frac{dy}{dx} = x^2(y-2)(y+1),$$

show that $\dfrac{y-2}{y+1} = Ae^{x^3}$, where A is a constant. **[5]**