1 Fig. 7 illustrates the growth of a population with time. The proportion of the ultimate (long term) population is denoted by x, and the time in years by t. When $t = 0$, $x = 0.5$, and as t increases, x approaches 1.

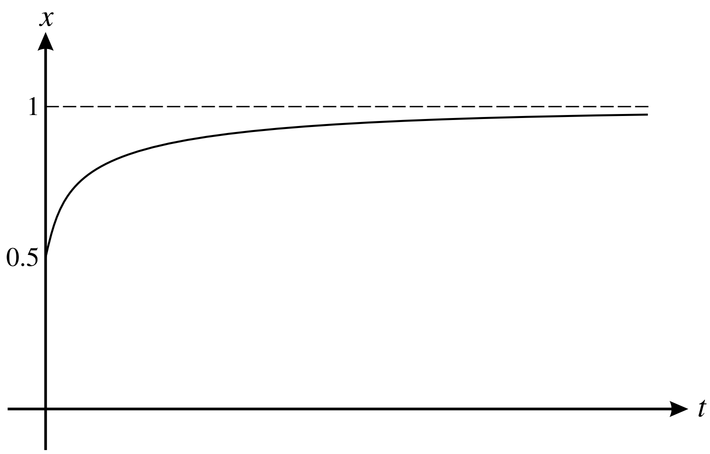

Fig. 7

One model for this situation is given by the differential equation

$$\frac{dx}{dt} = x(1-x).$$

(i) Verify that $x = \dfrac{1}{1+e^{-t}}$ satisfies this differential equation, including the initial condition. **[6]**

(ii) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. **[3]**

An alternative model for this situation is given by the differential equation

$$\frac{dx}{dt} = x^2(1-x),$$

with $x = 0.5$ when $t = 0$ as before.

(iii) Find constants A, B and C such that $\dfrac{1}{x^2(1-x)} = \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{1-x}$. **[4]**

(iv) Hence show that $t = 2 + \ln\left(\dfrac{x}{1-x}\right) - \dfrac{1}{x}$. **[5]**

(v) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. **[2]**

2 Scientists can estimate the time elapsed since an animal died by measuring its body temperature.

(i) Assuming the temperature goes down at a constant rate of 1.5 degrees Fahrenheit per hour, estimate how long it will take for the temperature to drop

(A) from 98 °F to 89 °F,

(B) from 98 °F to 80 °F. **[2]**

In practice, rate of temperature loss is not likely to be constant. A better model is provided by Newton's law of cooling, which states that the temperature θ in degrees Fahrenheit t hours after death is given by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_0),$$

where θ_0 °F is the air temperature and k is a constant.

(ii) Show by integration that the solution of this equation is $\theta = \theta_0 + A\mathrm{e}^{-kt}$, where A is a constant. **[5]**

The value of θ_0 is 50, and the initial value of θ is 98. The initial rate of temperature loss is 1.5 °F per hour.

(iii) Find A, and show that $k = 0.031\,25$. **[4]**

(iv) Use this model to calculate how long it will take for the temperature to drop

(A) from 98 °F to 89 °F,

(B) from 98 °F to 80 °F. **[5]**

(v) Comment on the results obtained in parts **(i)** and **(iv)**. **[1]**

3 Some years ago an island was populated by red squirrels and there were no grey squirrels. Then grey squirrels were introduced.

The population x, in thousands, of red squirrels is modelled by the equation

$$x = \frac{a}{1+kt},$$

where t is the time in years, and a and k are constants. When $t = 0, x = 2.5$.

(i) Show that $\frac{dx}{dt} = -\frac{kx^2}{a}$. [3]

(ii) Given that the initial population of 2.5 thousand red squirrels reduces to 1.6 thousand after one year, calculate a and k. [3]

(iii) What is the long-term population of red squirrels predicted by this model? [1]

The population y, in thousands, of grey squirrels is modelled by the differential equation

$$\frac{dy}{dt} = 2y - y^2.$$

When $t = 0, y = 1$.

(iv) Express $\frac{1}{2y - y^2}$ in partial fractions. [4]

(v) Hence show by integration that $\ln\left(\frac{y}{2 - y}\right) = 2t$.

Show that $y = \frac{2}{1 + e^{-2t}}$. [7]

(vi) What is the long-term population of grey squirrels predicted by this model? [1]

4 A curve has equation

$$x^2 + 4y^2 = k^2,$$

where k is a positive constant.

(i) Verify that

$$x = k\cos\theta, \qquad y = \tfrac{1}{2}k\sin\theta,$$

are parametric equations for the curve. [3]

(ii) Hence or otherwise show that $\dfrac{dy}{dx} = -\dfrac{x}{4y}$. [3]

(iii) Fig. 8 illustrates the curve for a particular value of k. Write down this value of k. [1]

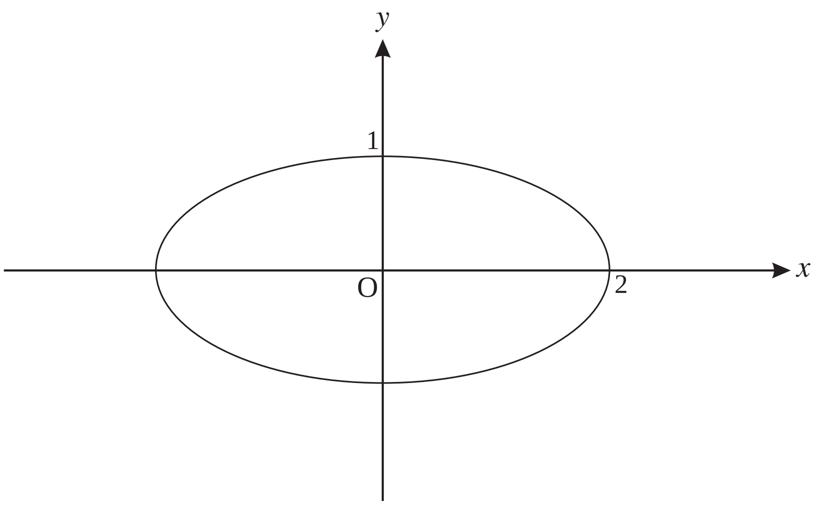

Fig. 8

(iv) Copy Fig. 8 and on the same axes sketch the curves for $k = 1$, $k = 3$ and $k = 4$. [3]

On a map, the curves represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(v) Explain why the path of the stream is modelled by the differential equation

$$\frac{dy}{dx} = \frac{4y}{x}.$$ [2]

(vi) Solve this differential equation.

Given that the path of the stream passes through the point $(2, 1)$, show that its equation is $y = \dfrac{x^4}{16}$. [6]