1 In a chemical process, the mass M grams of a chemical at time t minutes is modelled by the differential equation

$$\frac{dM}{dt} = \frac{M}{t(1+t^2)}.$$

(i) Find $\int \frac{t}{1+t^2}\,dt$. [3]

(ii) Find constants A, B and C such that

$$\frac{1}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{1+t^2}.$$ [5]

(iii) Use integration, together with your results in parts (i) and (ii), to show that

$$M = \frac{Kt}{\sqrt{1+t^2}},$$

where K is a constant. [6]

(iv) When $t = 1$, $M = 25$. Calculate K.

What is the mass of the chemical in the long term? [4]

2 The growth of a tree is modelled by the differential equation

$$10\frac{\mathrm{d}h}{\mathrm{d}t} = 20 - h,$$

where h is its height in metres and the time t is in years. It is assumed that the tree is grown from seed, so that $h = 0$ when $t = 0$.

(i) Write down the value of h for which $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, and interpret this in terms of the growth of the tree. **[1]**

(ii) Verify that $h = 20\left(1 - \mathrm{e}^{-0.1t}\right)$ satisfies this differential equation and its initial condition. **[5]**

The alternative differential equation

$$200\frac{\mathrm{d}h}{\mathrm{d}t} = 400 - h^2$$

is proposed to model the growth of the tree. As before, $h = 0$ when $t = 0$.

(iii) Using partial fractions, show by integration that the solution to the alternative differential equation is

$$h = \frac{20\left(1 - \mathrm{e}^{-0.2t}\right)}{1 + \mathrm{e}^{-0.2t}}.$$

[9]

(iv) What does this solution indicate about the long-term height of the tree? **[1]**

(v) After a year, the tree has grown to a height of 2 m. Which model fits this information better? **[3]**

3

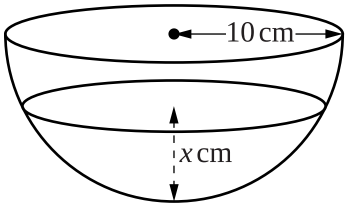

Fig. 9

Fig. 9 shows a hemispherical bowl, of radius 10 cm, filled with water to a depth of x cm. It can be shown that the volume of water, V cm^3, is given by

$$V = \pi(10x^2 - \tfrac{1}{3}x^3).$$

Water is poured into a leaking hemispherical bowl of radius 10 cm. Initially, the bowl is empty. After t seconds, the volume of water is changing at a rate, in cm^3 s^{-1}, given by the equation

$$\frac{dV}{dt} = k(20 - x),$$

where k is a constant.

(i) Find $\frac{dV}{dx}$, and hence show that $\pi x \frac{dx}{dt} = k$. **[4]**

(ii) Solve this differential equation, and hence show that the bowl fills completely after T seconds, where $T = \frac{50\pi}{k}$. **[5]**

Once the bowl is full, the supply of water to the bowl is switched off, and water then leaks out at a rate of kx cm^3 s^{-1}.

(iii) Show that, t seconds later, $\pi(20 - x)\frac{dx}{dt} = -k$. **[3]**

(iv) Solve this differential equation.

Hence show that the bowl empties in $3T$ seconds. **[6]**

4 A particle is moving vertically downwards in a liquid. Initially its velocity is zero, and after t seconds it is $v\,\mathrm{m\,s^{-1}}$. Its terminal (long-term) velocity is $5\,\mathrm{m\,s^{-1}}$.

A model of the particle's motion is proposed. In this model, $v = 5\left(1 - \mathrm{e}^{-2t}\right)$.

(i) Show that this equation is consistent with the initial and terminal velocities. Calculate the velocity after 0.5 seconds as given by this model. **[3]**

(ii) Verify that v satisfies the differential equation $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 10 - 2v$. **[3]**

In a second model, v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.4v^2.$$

As before, when $t = 0$, $v = 0$.

(iii) Show that this differential equation may be written as

$$\frac{10}{(5-v)(5+v)}\frac{\mathrm{d}v}{\mathrm{d}t} = 4.$$

Using partial fractions, solve this differential equation to show that

$$t = \tfrac{1}{4}\ln\left(\frac{5+v}{5-v}\right).$$ **[8]**

This can be re-arranged to give $v = \dfrac{5\left(1 - \mathrm{e}^{-4t}\right)}{1 + \mathrm{e}^{-4t}}$. [You are **not** required to show this result.]

(iv) Verify that this model also gives a terminal velocity of $5\,\mathrm{m\,s^{-1}}$.

Calculate the velocity after 0.5 seconds as given by this model. **[3]**

The velocity of the particle after 0.5 seconds is measured as $3\,\mathrm{m\,s^{-1}}$.

(v) Which of the two models fits the data better? **[1]**