1 A drug is administered by an intravenous drip. The concentration, x, of the drug in the blood is measured as a fraction of its maximum level. The drug concentration after t hours is modelled by the differential equation

$$\frac{dx}{dt} = k(1+x-2x^2),$$

where $0 \leqslant x < 1$, and k is a positive constant. Initially, $x = 0$.

(i) Express $\frac{1}{(1+2x)(1-x)}$ in partial fractions. **[3]**

(ii) Hence solve the differential equation to show that $\frac{1+2x}{1-x} = e^{3kt}$. **[7]**

(iii) After 1 hour the drug concentration reaches 75% of its maximum value and so $x = 0.75$.

Find the value of k, and the time taken for the drug concentration to reach 90% of its maximum value. **[3]**

(iv) Rearrange the equation in part **(ii)** to show that $x = \frac{1-e^{-3kt}}{1+2e^{-3kt}}$.

Verify that in the long term the drug concentration approaches its maximum value. **[5]**

2 A curve has parametric equations $x = e^{3t}, y = te^{2t}$.

(i) Find $\frac{dy}{dx}$ in terms of t. Hence find the exact gradient of the curve at the point with parameter $t = 1$. **[4]**

(ii) Find the cartesian equation of the curve in the form $y = ax^b \ln x$, where a and b are constants to be determined. **[3]**

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3 Fig. 8.1 shows an upright cylindrical barrel containing water. The water is leaking out of a hole in the side of the barrel.

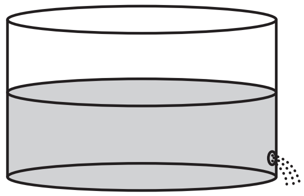

Fig. 8.1

The height of the water surface above the hole t seconds after opening the hole is h metres, where

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -A\sqrt{h}$$

and where A is a positive constant. Initially the water surface is 1 metre above the hole.

(i) Verify that the solution to this differential equation is

$$h = \left(1 - \tfrac{1}{2}At\right)^2.$$ **[3]**

The water stops leaking when $h = 0$. This occurs after 20 seconds.

(ii) Find the value of A, and the time when the height of the water surface above the hole is 0.5 m. **[4]**

Fig. 8.2 shows a similar situation with a different barrel; h is in metres.

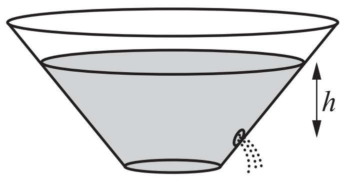

Fig. 8.2

For this barrel,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -B\frac{\sqrt{h}}{(1+h)^2},$$

where B is a positive constant. When $t = 0$, $h = 1$.

(iii) Solve this differential equation, and hence show that

$$h^{\frac{1}{2}}(30 + 20h + 6h^2) = 56 - 15Bt.$$ **[7]**

(iv) Given that $h = 0$ when $t = 20$, find B.

Find also the time when the height of the water surface above the hole is 0.5 m. **[4]**

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4 The motion of a particle is modelled by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} + 4x = 0,$$

where x is its displacement from a fixed point, and v is its velocity.

Initially $x = 1$ and $v = 4$.

(i) Solve the differential equation to show that $v^2 = 20 - 4x^2$. **[4]**

Now consider motion for which $x = \cos 2t + 2\sin 2t$, where x is the displacement from a fixed point at time t.

(ii) Verify that, when $t = 0$, $x = 1$. Use the fact that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ to verify that when $t = 0$, $v = 4$. **[4]**

(iii) Express x in the form $R\cos(2t - \alpha)$, where R and α are constants to be determined, and obtain the corresponding expression for v. Hence or otherwise verify that, for this motion too, $v^2 = 20 - 4x^2$. **[7]**

(iv) Use your answers to part **(iii)** to find the maximum value of x, and the earliest time at which x reaches this maximum value. **[3]**

5 The total value of the sales made by a new company in the first t years of its existence is denoted by £V. A model is proposed in which the rate of increase of V is proportional to the square root of V. The constant of proportionality is k.

(i) Express the model as a differential equation.

Verify by differentiation that $V = (\frac{1}{2}kt + c)^2$, where c is an arbitrary constant, satisfies this differential equation. **[4]**

(ii) The value of the company's sales in its first year is £10 000, and the total value of the sales in the first two years is £40 000. Find V in terms of t. **[4]**

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