1 Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

(a) Suppose that the number of cases, P thousand, after time t months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0, P = 1$.

(i) By considering the greatest and least values of $\sin t$, write down the greatest and least values of P predicted by this model. [2]

(ii) Verify that P satisfies the differential equation $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2} P^2 \cos t$. [5]

(b) An alternative model is proposed, with differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \tfrac{1}{2}(2P^2 - P)\cos t. \qquad (*)$$

As before, $P = 1$ when $t = 0$.

(i) Express $\dfrac{1}{P(2P-1)}$ in partial fractions. [4]

(ii) Solve the differential equation (*) to show that

$$\ln\left(\frac{2P \quad 1}{P}\right) = \tfrac{1}{2}\sin t. \qquad [5]$$

This equation can be rearranged to give $P = \dfrac{1}{2 \quad e^{\frac{1}{2}\sin t}}$.

(iii) Find the greatest and least values of P predicted by this model. [4]

2 A particle is moving vertically downwards in a liquid. Initially its velocity is zero, and after t seconds it is $v\,\mathrm{m\,s^{-1}}$. Its terminal (long-term) velocity is $5\,\mathrm{m\,s^{-1}}$.

A model of the particle's motion is proposed. In this model, $v = 5\left(1 - \mathrm{e}^{-2t}\right)$.

(i) Show that this equation is consistent with the initial and terminal velocities. Calculate the velocity after 0.5 seconds as given by this model. **[3]**

(ii) Verify that v satisfies the differential equation $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 10 - 2v$. **[3]**

In a second model, v satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.4v^2.$$

As before, when $t = 0$, $v = 0$.

(iii) Show that this differential equation may be written as

$$\frac{10}{(5-v)(5+v)}\frac{\mathrm{d}v}{\mathrm{d}t} = 4.$$

Using partial fractions, solve this differential equation to show that

$$t = \tfrac{1}{4}\ln\left(\frac{5+v}{5-v}\right).$$ **[8]**

This can be re-arranged to give $v = \dfrac{5\left(1 - \mathrm{e}^{-4t}\right)}{1 + \mathrm{e}^{-4t}}$. [You are **not** required to show this result.]

(iv) Verify that this model also gives a terminal velocity of $5\,\mathrm{m\,s^{-1}}$.

Calculate the velocity after 0.5 seconds as given by this model. **[3]**

The velocity of the particle after 0.5 seconds is measured as $3\,\mathrm{m\,s^{-1}}$.

(v) Which of the two models fits the data better? **[1]**

3 Some years ago an island was populated by red squirrels and there were no grey squirrels. Then grey squirrels were introduced.

The population x, in thousands, of red squirrels is modelled by the equation

$$x = \frac{a}{1 + kt},$$

where t is the time in years, and a and k are constants. When $t = 0, x = 2.5$.

(i) Show that $\frac{dx}{dt} = -\frac{kx^2}{a}$. [3]

(ii) Given that the initial population of 2.5 thousand red squirrels reduces to 1.6 thousand after one year, calculate a and k. [3]

(iii) What is the long-term population of red squirrels predicted by this model? [1]

The population y, in thousands, of grey squirrels is modelled by the differential equation

$$\frac{dy}{dt} = 2y - y^2.$$

When $t = 0, y = 1$.

(iv) Express $\frac{1}{2y - y^2}$ in partial fractions. [4]

(v) Hence show by integration that $\ln\left(\frac{y}{2 \quad y}\right) = 2t$.

Show that $y = \frac{2}{1 + e^{-2t}}$. [7]

(vi) What is the long-term population of grey squirrels predicted by this model? [1]