1. Using the substitution $u = \cos x + 1$, or otherwise, show that

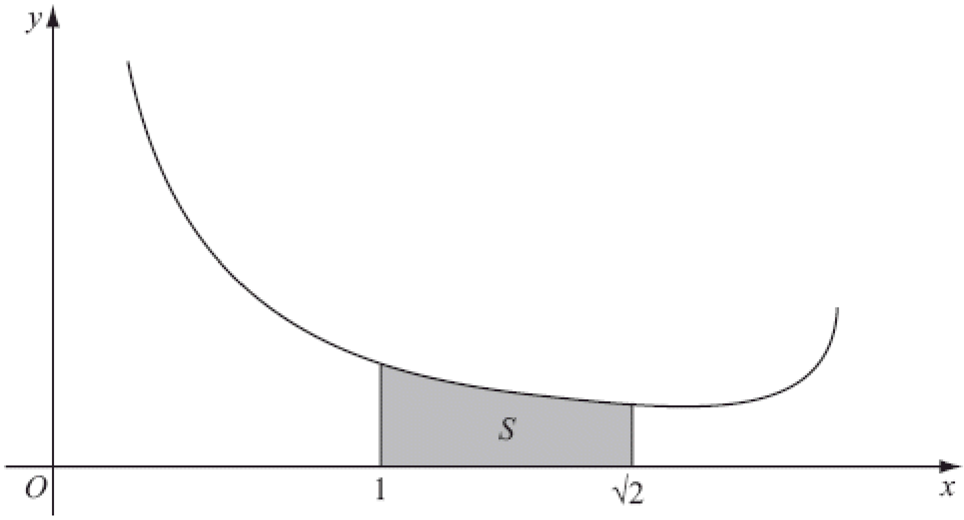
$$\int_0^{\frac{\pi}{2}} e^{\cos x+1} \sin x \, dx \; e(e - 1)$$

(Total 6 marks)

2. (a) Using the substitution $x = 2 \cos u$, or otherwise, find the exact value of

$$\int_1^{\sqrt{2}} \frac{1}{x^2 \sqrt{(4-x^2)}} dx$$

(7)



The diagram above shows a sketch of part of the curve with equation $y = \dfrac{4}{x(4-x^2)^{\frac{1}{4}}}$,

$0 < x < 2$.

The shaded region S, shown in the diagram above, is bounded by the curve, the x-axis and the lines with equations $x = 1$ and $x = \sqrt{2}$. The shaded region S is rotated through 2π radians about the x-axis to form a solid of revolution.

(b) Using your answer to part (a), find the exact volume of the solid of revolution formed.

(3)
(Total 10 marks)

3. (a) Find $\int \tan^2 x \, dx$.

(2)

(b) Use integration by parts to find $\int \frac{1}{x^3} \ln x \, dx$.

(4)

(c) Use the substitution $u = 1 + e^x$ to show that

$$\int \frac{e^{3x}}{1+e^x} dx = \frac{1}{2}e^{2x} - e^x + \ln(1+e^x) + k,$$

where k is a constant.

(7)
(Total 13 marks)

4. Use the substitution $u = 2^x$ to find the exact value of

$$\int_0^1 \frac{2^x}{(2^x+1)^2} dx.$$

(Total 6 marks)

5. Using the substitution $u^2 = 2x - 1$, or otherwise, find the exact value of

$$\int_1^5 \frac{3x}{\sqrt{(2x-1)}} dx$$

(Total 8 marks)

6. Use the substitution $x = \sin\theta$ to find the exact value of

$$\int_0^{\frac{1}{2}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,\mathrm{d}x\,.$$

(Total 7 marks)

7. Use the substitution $u = 1 + \sin x$ and integration to show that

$$\int \sin x \cos x\,(1 + \sin x)^5\,\mathrm{d}x = \frac{1}{42}(1 + \sin x)^6\,[6\sin x - 1] + \text{constant}.$$

(Total 8 marks)

8. Use the substitution $u^2 = (x - 1)$ to find

$$\int \frac{x^2}{\sqrt{(x-1)}}\,\mathrm{d}x\,,$$

giving your answer in terms of x.

(Total 10 marks)

9. Use the substitution $u = 4 + 3x^2$ to find the exact value of

$$\int_0^2 \frac{2x}{(4+3x^2)^2}\,\mathrm{d}x\,.$$

(Total 6 marks)

1.	$\frac{du}{dx} = -\sin x$		B1	
	$\int \sin x\, e^{\cos x+1} dx = -\int e^u du$		M1 A1	
	$= -e^u$	ft sign error	A1 ft	
	$= -e^{\cos x+1}$			
	$\left[-e^{\cos x+1}\right]_0^{\frac{\pi}{2}} = -e^1 - \left(-e^2\right)$	or equivalent with u	M1	
	$= e\left(e-1\right)$ ∗	cso	A1	6

[6]

2. (a)	$\frac{dx}{du} = -2\sin u$		B1	
	$\int \frac{1}{x^2\sqrt{4-x^2}} dx = \int \frac{1}{(2\cos u)^2\sqrt{4-(2\cos u)^2}} \times -2\sin u\, du$		M1	
	$= \int \frac{-2\sin u}{4\cos^2 u\sqrt{4\sin^2 u}} du$	Use of $1 - \cos^2 u = \sin^2 u$	M1	
	$= -\frac{1}{4}\int \frac{1}{\cos^2 u} du$	$\pm k\int \frac{1}{\cos^2 u} du$	M1	
	$= -\frac{1}{4}\tan u (+C)$	$\pm k \tan u$	M1	
	$x = \sqrt{2} \Rightarrow \sqrt{2} = 2\cos u \Rightarrow u = \frac{\pi}{4}$			
	$x = 1 \Rightarrow 1 = 2\cos u \Rightarrow u = \frac{\pi}{3}$		M1	
	$\left[-\frac{1}{4}\tan u\right]_{\frac{\pi}{4}}^{\frac{\pi}{4}} = -\frac{1}{4}\left(\tan\frac{\pi}{4} - \tan\frac{\pi}{3}\right)$			
	$= -\frac{1}{4}\left(1-\sqrt{3}\right)\left(= \frac{\sqrt{3}-1}{4}\right)$		A1	7

(b) $V = \pi \int_1^{\sqrt{2}} \left(\frac{4}{x(4-x^2)^{\frac{1}{4}}} \right)^2 \mathrm{d}x$ M1

$= 16\pi \int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}} \mathrm{d}x$ $16\pi \times$ integral in (a) M1

$= 16\pi \left(\frac{\sqrt{3}-1}{4} \right)$ $16\pi \times$ their answer to part (a) A1ft 3

[10]

3. (a) $\int \tan^2 x \mathrm{d}x$

[*NB* : <u>$\sec^2 A = 1 + \tan^2 A$</u> gives <u>$\tan^2 A = \sec^2 A - 1$</u>] The correct <u>underlined identity</u>. M1 oe

$= \int \sec^2 x - 1 dx$

= <u>$\tan x - x (+c)$</u> Correct integration with/without + c A1 2

(b) $\int \frac{1}{x^3} \ln x \, \mathrm{d}x$

$\left\{ \begin{array}{ll} u = \ln x & \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = x^{-3} & \Rightarrow v = \frac{x^{-2}}{-2} = \frac{-1}{2x^2} \end{array} \right\}$

$= -\frac{1}{2x^2} \ln x - \int -\frac{1}{2x^2} \cdot \frac{1}{x} \mathrm{d}x$ Use of 'integration by parts' formula in the correct direction. M1

Correct direction means that $u = \ln x$.

Correct expression. A1

$= -\frac{1}{2x^2} \ln x + \frac{1}{2} \int \frac{1}{x^3} \mathrm{d}x$ An attempt to multiply through $\frac{k}{x^n}, n \in$ □ n . . .2 by $\frac{1}{x}$ and an attempt to ...

$= -\underline{\frac{1}{2x^2} \ln x + \frac{1}{2}\left(-\frac{1}{2x^2} \right)} (+c)$ ... "integrate"(process the result); M1

<u>correct solution</u> with/without + c A1 oe 4

(c) $\int \frac{e^{3x}}{1+e^x} dx$

$\left\{ u = 1 + e^x \Rightarrow \underline{\frac{du}{dx} = e^x}, \underline{\frac{dx}{du} = \frac{1}{e^x}}, \underline{\frac{dx}{du} = \frac{1}{u-1}} \right\}$ Differentiating to find any one of the three underlined B1

$\int \frac{e^{2x}.e^x}{1+e^x} dx = \int \frac{(u-1)^2.e^x}{u} . \frac{1}{e^x} du$ Attempt to substitute for e^{2x} = f(u), their $\frac{dx}{du} = \frac{1}{e^x}$ and $u = 1 + e^x$

or = $\int \frac{(u-1)^3}{u} . \frac{1}{(u-1)} du$ or e^{3x} =f(u), their $\frac{dx}{du} = \frac{1}{u-1}$ and u =1 + e^x. M1 *

= $\int \frac{(u-1)^2}{u} du$ $\underline{\int \frac{(u-1)^2}{u} du}$ A1

= $\int \frac{u^2 - 2u + 1}{u} du$ An attempt to multiply out their numerator to give at least three terms

= $\int u - 2 + \frac{1}{u} du$ and divide through each term by u dM1 *

= $\frac{u^2}{2} - 2u + 1\text{n}u\ (+c)$ Correct integration with/without +c A1

= $\frac{(1+e^x)^2}{2} - 2(1+e^x) + 1\text{n}(1+e^x) + c$ Substitutes u = 1 + e^x back into their integrated expression with at least two terms. dM1 *

= $\frac{1}{2} + e^x + \frac{1}{2}e^{2x} - 2 - 2e^x + 1\text{n}\ (1 + e^x) + c$

= $\frac{1}{2} + e^x + \frac{1}{2}e^{2x} - 2 - 2e^x + \ln(1 + e^x) + c$

= $\frac{1}{2}e^{2x} - e^x + \ln(1 + e^x) - \frac{3}{2} + c$

= $\frac{1}{2}e^{2x} - e^x + \ln(1 + e^x) + k$ **AG** $\underline{\frac{1}{2}e^{2x} - e^x + 1\text{n}(1+e^x) + k}$ must use a + c + and "$-\frac{3}{2}$" combined. A1 cso 7

[13]

4. $\int_0^1 \frac{2^x}{(2^x+1)^2}\,\mathrm{d}x$, with substitution $u = 2^x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2^x.\ln 2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2^x.\ln 2}$

$\int \frac{2^x}{(2^x+1)^2}\,\mathrm{d}x = \left(\frac{1}{\ln 2}\right)\int \frac{1}{(u+1)^2}\,\mathrm{d}u$ 6

$= \left(\frac{1}{\ln 2}\right)\left(\frac{-1}{(u+1)}\right) + c$

B1 $\frac{\mathrm{d}u}{\mathrm{d}x} = 2^x.\ln 2$ or $\frac{\mathrm{d}u}{\mathrm{d}x} = u.\ln 2$ or $\left(\frac{1}{u}\right)\frac{\mathrm{d}u}{\mathrm{d}x} = \ln 2$

M1* $k\int \frac{1}{(u+1)^2}\,\mathrm{d}u$ where k is constant

M1 $(u + 1)^{-2} \to a(u + 1)^{-1}$ (*)

A1 $(u + 1)^{-2} \to -1.(u + 1)^{-1}$ (*)

(*) If you see this integration applied anywhere in a candidate's working then you can award M1, A1

change limits: when $x = 0$ & $x = 1$ then $u = 1$ & $u = 2$

$\int_0^1 \frac{2^x}{(2^x+1)^2}\,\mathrm{d}x = \frac{1}{\ln 2}\left[\frac{-1}{(u+1)}\right]_1^2$

$= \frac{1}{\ln 2}\left[\left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right)\right]$

$= \frac{1}{6\ln 2}$

depM1 Correct use of limits $u = 1$ and $u = 2$

A1aef $\frac{1}{6\ln 2}$ or $\frac{1}{\ln 4} - \frac{1}{\ln 8}$ or $\frac{1}{2\ln 2} - \frac{1}{3\ln 2}$ (*)

Exact value only!

(*) There are other acceptable answers for A1. eg: $\frac{1}{2\ln 8}$ or $\frac{1}{\ln 64}$

NB: Use your calculator to check eg. 0.240449...

Alternatively candidate can revert back to x...

$$\int_0^1 \frac{2^x}{(2^x+1)^2}\,dx = \frac{1}{\ln 2}\left[\frac{-1}{(2^x+1)}\right]_0^1$$

$$= \frac{1}{\ln 2}\left[\left(-\frac{1}{3}\right)-\left(-\frac{1}{2}\right)\right]$$

$$= \underline{\frac{1}{6\ln 2}}$$

depM1* Correct use of limits $x = 0$ and $x = 1$

A1aef $\underline{\frac{1}{6\ln 2}}$ or $\underline{\frac{1}{\ln 4}-\frac{1}{\ln 8}}$ or $\underline{\frac{1}{2\ln 2}-\frac{1}{3\ln 2}}$ (*)

Exact value only!

(*) There are other acceptable answers for A1. eg: $\underline{\frac{1}{2\ln 8}}$ or $\underline{\frac{1}{\ln 64}}$

NB: Use your calculator to check eg. 0.240449...

[6]

5. Uses substitution to obtain $x = \text{f}(u)$ $\left[\frac{u^2+1}{2}\right]$, M1

and to obtain $u\frac{du}{dx}$ = const. or equiv. M1

Reaches $\int \frac{3(u^2+1)}{2u}\,u\,du$ or equivalent A1

Simplifies integrand to $\int\left(3u^2+\frac{3}{2}\right)$ or equiv. M1

Integrates to $\frac{1}{2}u^2+\frac{3}{2}u$

A1ft dependent on all previous Ms

Uses new limits 3 and 1 substituting and subtracting

(or returning to function of x with old limits) M1

To give 16 cso A1 8

[8]

"By Parts"

Attempt at " right direction" by parts — M1

$$[\ 3x\ (2x-1)^{\frac{1}{2}} - \{\int 3(2x-1)^{\frac{3}{2}}\,dx\}]$$ M1{M1A1}

$$\ldots\ldots\ldots\ldots\ldots \quad - \quad (2x-1)^{\frac{3}{2}}$$ M1A1ft

Uses limits 5 and 1 correctly; [42 – 26] 16 — M1A1

6. $\int \frac{1}{(x-x^2)^{\frac{1}{2}}}\,dx = \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}} \cos\theta\,d\theta$ — M1

Use of $x = \sin\theta$ and $\frac{dx}{d\theta} = \cos\theta$

$= \int \frac{1}{\cos^2\theta}\,d\theta$ — M1 A1

$= \int \sec^2\theta\,d\theta = \tan\theta$ — M1 A1

Using the limits 0 and $\frac{\pi}{6}$ to evaluate integral — M1

$[\tan\theta]_0^{\frac{\pi}{6}} = \frac{1}{\sqrt{3}}\left(= \frac{\sqrt{3}}{3}\right)$ — cao A1

Alternative for final M1 A1

Returning to the variable x and using the limits 0 and $\frac{1}{2}$ to evaluate integral — M1

$\left[\frac{x}{\sqrt{(1-x^2)}}\right]_0^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\left(= \frac{\sqrt{3}}{3}\right)$ — cao A1

[7]

7. $u = 1 + \sin x \Rightarrow \frac{du}{dx} = \cos x$ or $du = \cos x\,dx$ or $dx = \frac{du}{\cos x}$ — M1

$I = \int (u-1)u^5\,du$ — M1, A1

Full sub. to I in terms of u, correct

$= \int (u^6 - u^5)\,du$ — M1

Correct split

$= \frac{u^7}{7} - \frac{u^6}{6}$ (+ c) M1, A1

M1 for $u^n \to u^{n+1}$

$= \frac{u^6}{42}(6u - 7)$ (+ c) M1

Attempt to factorise

$= \frac{(1+\sin x)^6}{42}(6 \sin x + 6 - 7) + c \to \underline{\frac{(1+\sin x)^6}{42}}\underline{(6 \sin x - 1)\ (+ c)}$ (*) A1 cso

[8]

Alt: Integration by parts M1

$I = (u-1)\frac{u^6}{6} - \frac{1}{6}\int u^6 \, du$ Attempt first stage M1

$= (u-1)\frac{u^6}{6} - \frac{u^7}{42}$ Full integration A1

$(= \frac{u^7}{6} - \frac{u^6}{6} - \frac{u^7}{42}$ or $\frac{6u^7 - 7u^6}{42})$

rest as scheme

8. $u^2 = x - 1$

$2u \frac{du}{dx} = 1 \quad x = u^2 + 1$ M1 A1 A1

$I = \int \frac{(u^2+1)^2}{u} 2u \, du$

$= 2\int (u^4 + 2u^2 + 1) \, du$ A1

$= 2\left[\frac{u^5}{5} + \frac{2u^3}{3} + u\right] + c$ M1 A1

$= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + c$ M1 A1 10

[10]

9.	Uses $\frac{du}{dx} = 6x$	M1	
	To give $\int \frac{1}{u^2} \frac{du}{3}$	A1	
	Integrates to give $-\frac{1}{3u}$	M1, A1	
	Uses correct limits 16 and 4 (or 2 and 0 for x)	M1	
	To obtain $-\frac{1}{48} + \frac{1}{12} = \frac{1}{16}$	A1	6

[6]

1. This question was generally well done and, helped by the printed answer, many produced fully correct answers. The commonest error was to omit the negative sign when differentiating $\cos x + 1$. The order of the limits gave some difficulty. Instead of the correct $-\int_1^2 e^u \, du$, an incorrect version $-\int_1^2 e^u \, du$ was produced and the resulting expressions manipulated to the printed result and working like $-(e^2 - e^1) = -e^2 + e^1 = e(e - 1)$ was not uncommon.

Some candidates got into serious difficulties when, through incorrect algebraic manipulation, they obtained $-\int e^u \sin^2 x \, du$ instead of $-\int e^u \, du$. This led to expressions such as $\int e^u (u^2 - 2u) \, du$ and the efforts to integrate this, either by parts twice or a further substitution, often ran to several supplementary sheets. The time lost here inevitably led to difficulties in finishing the paper. Candidates need to have some idea of the amount of work and time appropriate to a 6 mark question and, if they find themselves exceeding this, realise that they have probably made a mistake and that they would be well advised to go on to another question.

2. Answers to part (a) were mixed, although most candidates gained some method marks. A surprisingly large number of candidates failed to deal with $\sqrt{4 - 4\cos^2 u}$ correctly and many did not recognise that $\int \frac{1}{\cos^2 x} dx = \int \sec^2 x \, dx = \tan x (+C)$ in this context. Nearly all converted the limits correctly. Answers to part (b) were also mixed. Some could not get beyond stating the formula for the volume of revolution while others gained the first mark, by substituting the equation given in part (b) into this formula, but could not see the connection with part (a). Candidates could recover here and gain full follow through marks in part (b) after an incorrect attempt at part (a).

3. In part (a), a surprisingly large number of candidates did not know how to integrate $\tan^2 x$. Examiners were confronted with some strange attempts involving either double angle formulae or logarithmic answers such as $\ln(\sec^2 x)$ or $\ln(\sec^4 x)$. Those candidates who realised that the needed the identity $\sec^2 x = 1 + \tan^2 x$ sometimes wrote it down incorrectly.

Part (b) was probably the best attempted of the three parts in the question. This was a tricky integration by parts question owing to the term of $\frac{1}{x^3}$, meaning that candidates had to be especially careful when using negative powers. Many candidates applied the integration by parts formula correctly and then went on to integrate an expression of the form $\frac{k}{x^3}$ to gain 3 out of the 4 marks available. A significant number of candidates failed to gain the final accuracy mark owing to sign errors or errors with the constants α and β in $\frac{\alpha}{x^2}\ln x + \frac{\beta}{x^2} + c$. A minority of candidates applied the by parts formula in the 'wrong direction' and incorrectly stated that $\frac{dv}{dh} = \ln x$ implied $v = \frac{1}{x}$.

In part (c), most candidates correctly differentiated the substitution to gain the first mark. A significant proportion of candidates found the substitution to obtain an integral in terms of u more demanding. Some candidates did not realise that e^{2x} and e^{3x} are $(e^x)^2$ and $(e^x)^3$ respectively and hence $u^2 - 1$, rather than $(u - 1)^2$ was a frequently encountered error seen in the numerator of the substituted expression. Fewer than half of the candidates simplified their substituted expression to arrive at the correct result of $\int \frac{(u-1)^2}{u} \mathrm{d}u$. Some candidates could not proceed further at this point but the majority of the candidates who achieved this result were able to multiply out the numerator, divide by u, integrate and substitute back for u. At this point some candidates struggled to achieve the expression required. The most common misconception was that the constant of integration was a fixed constant to be determined, and so many candidates concluded that $k = -\frac{3}{2}$. Many candidates did not realise that $-\frac{3}{2}$ when added to c combined to make another arbitrary constant k.

4. Many candidates had difficulties with the differentiation of the function $u = 2^x$, despite the same problem being posed in the January 2007 paper, with incorrect derivatives of $\frac{\mathrm{d}u}{\mathrm{d}x} = 2^x$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = x\,2^{x-1}$ being common. Those candidates who differentiated u with respect to x to obtain either $2^x \ln 2$ or 2^x often failed to replace 2^x with u; or if they did this, they failed to cancel the variable u from the numerator and the denominator of their algebraic fraction. Therefore, at this point candidates proceeded to do some "very complicated" integration, always with no chance of a correct solution.

Those candidates who attempted to integrate $k(u + 1)^{-2}$ usually did this correctly, but there were a significant number of candidates who either integrated this incorrectly to give $k(u + 1)^{-3}$ or $\ln f(u)$.

There were a significant proportion of candidates who proceeded to integrate $u(u + 1)^{-2}$ with respect to x and did so by either treating the leading u as a constant or using integration by parts.

Many candidates correctly changed the limits from 0 and 1 to 1 and 2 to obtain their final answer. Some candidates instead substituted u for 2^x and used limits of 0 and 1.

5. Most candidates chose to use the given substitution but the answers to this question were quite variable. There were many candidates who gave succinct, neat, totally correct solutions, and generally if the first two method marks in the scheme were gained good solutions usually followed.

The biggest problem, as usual, was in the treatment of "dx": those who differentiated $u^2 = 2x - 1$ implicitly were usually more successful, in their subsequent manipulation, than those who chose to write $u = \sqrt{(2x-1)}$ and then find $\frac{\mathrm{d}u}{\mathrm{d}x}$; many candidates ignored the dx altogether, or effectively treated it as du, and weaker candidates often "integrated" an expression involving both x and u terms. Some candidates spoilt otherwise good solutions by applying the wrong limits.

6. This was a question which candidates tended to either get completely correct or score very few marks. If the dx is ignored when substituting, an integral is obtained which is extremely difficult to integrate at this level and little progress can be made. Those who knew how to deal with the dx often completed correctly. A few, on obtaining $\tan\theta$, substituted 0 and $\frac{1}{2}$ instead of 0 and $\frac{\pi}{6}$. Very few attempted to return to the original variable and those who did were rarely successful.

7. Most candidates were able to make a start here but a number did not progress much beyond $\frac{du}{dx} = \cos x$. Some failed to realize that $\sin x = (1 - u)$ and others tried integrating an expression with a mixture of terms in x and u. Those who reached $\int (u-1)u^5 \, du$ usually went on to score the first 6 marks, but the final two marks were only scored by the most able who were able to complete the factorization and simplification clearly and accurately.

8. No Report available for this question.

9. No Report available for this question.