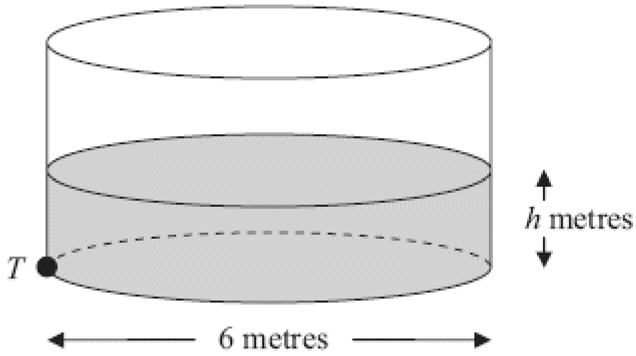
1.



The diagram above shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that t minutes after the tap has been opened

$$75\frac{\mathrm{d}h}{\mathrm{d}t} = (4 - 5h)$$

(5)

When $t = 0$, $h = 0.2$

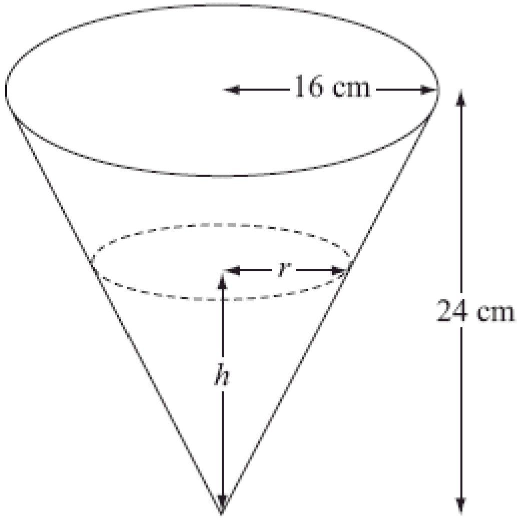
(b) Find the value of t when $h = 0.5$

(6)
(Total 11 marks)

2. The area A of a circle is increasing at a constant rate of 1.5 cm^2 s^{-1}. Find, to 3 significant figures, the rate at which the radius r of the circle is increasing when the area of the circle is 2 cm^2.

(Total 5 marks)

3.



A container is made in the shape of a hollow inverted right circular cone. The height of the container is 24 cm and the radius is 16 cm, as shown in the diagram above. Water is flowing into the container. When the height of water is h cm, the surface of the water has radius r cm and the volume of water is V cm^3.

(a) Show that $V = \frac{4\pi h^3}{27}$.

(2)

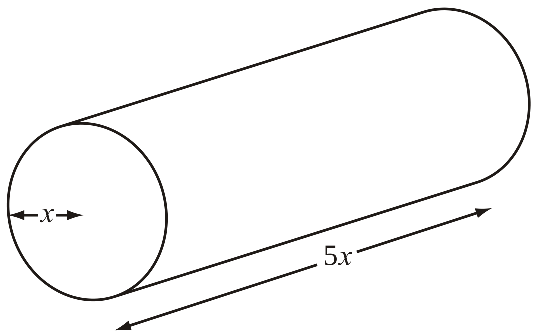
[*The volume V of a right circular cone with vertical height h and base radius r is given by the formula* $V = \frac{1}{3}\pi r^2 h$.]

Water flows into the container at a rate of 8 cm^3 s^{-1}.

(b) Find, in terms of π, the rate of change of h when h = 12.

(5)
(Total 7 marks)

4.



The diagram above shows a right circular cylindrical metal rod which is expanding as it is heated. After t seconds the radius of the rod is x cm and the length of the rod is $5x$ cm. The cross-sectional area of the rod is increasing at the constant rate of 0.032 $\text{cm}^2\ \text{s}^{-1}$.

(a) Find $\frac{\mathrm{d}x}{\mathrm{d}t}$ when the radius of the rod is 2 cm, giving your answer to 3 significant figures.

(4)

(b) Find the rate of increase of the volume of the rod when $x = 2$.

(4)
(Total 8 marks)

5. The volume of a spherical balloon of radius r cm is V cm^3, where $V = \frac{4}{3}\pi r^3$.

(a) Find $\frac{\mathrm{d}V}{\mathrm{d}r}$

(1)

The volume of the balloon increases with time t seconds according to the formula

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1000}{(2t+1)^2}, \quad t \geq 0.$$

(b) Using the chain rule, or otherwise, find an expression in terms of r and t for $\frac{\mathrm{d}r}{\mathrm{d}t}$.

(2)

(c) Given that $V = 0$ when $t = 0$, solve the differential equation $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1000}{(2t+1)^2}$, to obtain V in terms of t.

(4)

(d) Hence, at time $t = 5$,

(i) find the radius of the balloon, giving your answer to 3 significant figures,

(3)

(ii) show that the rate of increase of the radius of the balloon is approximately 2.90×10^{-2} cm s^{-1}.

(2)

(Total 12 marks)

6. The value £V of a car t years after the 1st January 2001 is given by the formula

$$V = 10\ 000 \times (1.5)^{-t}.$$

(a) Find the value of the car on 1st January 2005.

(2)

(b) Find the value of $\frac{\mathrm{d}V}{\mathrm{d}t}$ when $t = 4$.

(3)

(c) Explain what the answer to part (b) represents.

(1)

(Total 6 marks)

7. A drop of oil is modelled as a circle of radius r. At time t

$$r = 4(1 - e^{-\lambda t}), \qquad t > 0,$$

where λ is a positive constant.

(a) Show that the area A of the circle satisfies

$$\frac{dA}{dt} = 32\pi\lambda(e^{-\lambda t} - e^{-2\lambda t}).$$

(5)

In an alternative model of the drop of oil its area A at time t satisfies

$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}, \qquad t > 0.$$

Given that the area of the drop is 1 at $t = 1$,

(b) find an expression for A in terms of t for this alternative model.

(7)

(c) Show that, in the alternative model, the value of A cannot exceed 4.

(1)
(Total 13 marks)

1. (a)	$\frac{dV}{dt} = 0.48\pi - 0.6\pi h$		M1 A1	
	$V = 9\pi h \quad \Rightarrow \frac{dV}{dt} = 9\pi \frac{dh}{dt}$		B1	
	$9\pi \frac{dh}{dt} = 0.48\pi - 0.6\pi h$		M1	
	Leading to $75\frac{dh}{dt} = 4 - 5h$ ∗	cso	A1	5
(b)	$\int \frac{75}{4-5h} dh = \int 1 dt$	separating variables	M1	
	$-15\ln(4-5h) = t \ (+C)$		M1 A1	
	$-15\ln(4-5h) = t + C$			
	When $t = 0$, $h = 0.2$			
	$-15\ln 3 = C$		M1	
	$t = 15\ln 3 - 15\ln(4-5h)$			
	When $h = 0.5$			
	$t = 15\ln 3 - 15\ln 1.5 = 15\ln\left(\frac{3}{1.5}\right) = 15\ln 2$	awrt 10.4	M1 A1	
	Alternative for last 3 marks			
	$t = [-15\ln(4-5h)]_{0.2}^{0.5}$			
	$= -15\ln 1.5 + 15\ln 3$		M1 M1	
	$= 15\ln\left(\frac{3}{1.5}\right) = 151n\ 2$	awrt 10.4	A1	6

[11]

2. $\frac{dA}{dt} = 1.5$ B1

$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$ B1

When $A = 2$

$2 = \pi r^2 \Rightarrow r = \sqrt{\frac{2}{\pi}} (= 0.797884...)$ M1

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$1.5 = 2\pi r \frac{dr}{dt}$ M1

$\frac{dr}{dt} = \frac{1.5}{2\pi\sqrt{\frac{2}{\pi}}} \approx 0.299$ awrt 0.299 A1

[5]

3. (a) Similar triangles $\Rightarrow \frac{r}{h} = \frac{16}{24} \Rightarrow r = \frac{2h}{3}$ Uses similar triangles, ratios or trigonometry to find either one of these two expressions oe. M1

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2h}{3}\right)^2 h = \frac{4\pi h^3}{27}$ **AG** Substitutes $r = \frac{2h}{3}$ into the formula for the volume of water V. A1 2

(b) From the question, $\frac{dV}{dt} = 8$ — $\frac{dV}{dt} = 8$ — B1

$\frac{dV}{dh} = \frac{12\pi h^2}{27} = \frac{4\pi h^2}{9}$ — $\frac{dV}{dh} = \frac{12\pi h^2}{27} = \frac{4\pi h^2}{9}$ — B1

$\frac{dh}{dt} = \frac{dV}{dt} \div \frac{dV}{dh} = 8 \times \frac{9}{4\pi h^2} = \frac{18}{\pi h^2}$ — Candidate's $\frac{dV}{dt} \div \frac{dV}{dh}$; — M1;

$8 \div \left(\frac{12\pi h^2}{27}\right)$ or $8 \times \frac{9}{4\pi h^2}$ or $\frac{18}{\pi h^2}$ oe — A1

When $h = 12$, $\frac{dh}{dt} = \frac{18}{144\pi} = \frac{1}{8\pi}$ — $\frac{18}{144\pi}$ or $\frac{1}{8\pi}$ — A1 oe isw — 5

Note the answer must be a one term exact value.
Note, also you can ignore subsequent working after $\frac{18}{144\pi}$

[7]

4. (a) From question, $\frac{dA}{dt} = 0.032$

$\left\{A = \pi x^2 \Rightarrow \frac{dA}{dx} =\right\} 2\pi x$

$\frac{dx}{dt} = \frac{dA}{dt} \div \frac{dA}{dx} = (0.032)\frac{1}{2\pi x}; \left\{= \frac{0.016}{\pi x}\right\}$

When $x = 2$ cm, $\frac{dx}{dt} = \frac{0.016}{2\pi}$

Hence, $\frac{dx}{dt} = 0.002546479...$ (cm s^{-1})

$\frac{dA}{dt} = 0.032$ seen or implied from working. — B1

$2\pi x$ by itself seen or implied from working — B1

$0.032 \div$ Candidate's $\frac{dA}{dx}$; — M1;

awrt 0.00255 — A1 **cso** — 4

(b) $V = \underline{\pi x^2(5x)} = \underline{5\pi x^3}$

$\frac{dV}{dx} = 15\pi x^2$

$\frac{dV}{dx} = \frac{dV}{dx} \times \frac{dx}{dt} = 15\pi x^2 . \left(\frac{0.016}{\pi x}\right); \{= 0.24x\}$

When $x = 2$ cm, $\frac{dV}{dt} = 0.24(2) = \underline{0.48}$ ($\text{cm}^3\ \text{s}^{-1}$)

$V = \underline{\pi x^2(5x)} = \underline{5\pi x^3}$	B1	
$\frac{dV}{dx} = 15\pi x^2$ or ft from candidate's V **in one variable**	B1ft	
Candidate's $\frac{dV}{dx} \times \frac{dx}{dt}$;	M1ft	
0.48 or awrt 0.48	A1 **cso**	4

[8]

5. (a) $\frac{dV}{dr} = 4\pi r^2$ B1 1

(b) Uses $\frac{dr}{dt} = \frac{dV}{dt} . \frac{dr}{dV}$ in any form, $= \frac{1000}{4\pi r^2 (2t+1)^2}$ M1, A1 2

(c) $V = \int 1000\,(2t + 1)^{-2}\,dt$ and integrate to $p(2t + 1)^{-1}$,

$= -500(2t + 1)^{-1}\ (+c)$ M1, A1

Using V = 0 when t = 0 to find c , (c = 500, or equivalent) M1

$\therefore V = 500(1 - \frac{1}{2t+1})$ (any form) A1 4

(d) (i) Substitute t = 5 to give V,

then use $r = \sqrt[3]{\left(\frac{3V}{4\pi}\right)}$ to give r , = *4.77* M1, A1 3

	(ii)	Substitutes t = 5 and r = 'their value' into 'their' part (b)	M1	
		$\frac{dr}{dt} = 0.0289$ (≈ 2.90×10^{-2}) (cm/s) AG	A1	2

[12]

6.	(a)	Substitutes $t = 4$ to give V, = 1975.31 or 1975.30 or 1975 or 1980 (3 s.f)	M1, A1	2
	(b)	$\frac{dV}{dt} = -\ln 1.5 \times V$; = –800.92 or –800.9 or –801	M1 A1; A1	3
		M1 needs ln 1.5 term		
	(c)	rate of decrease in value on 1st January 2005	B1	1

[6]

7.	(a)	$A = \pi r^2$, $\frac{dr}{dt} = 4\lambda e^{-\lambda t}$	B1, B1	
		$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, $\Rightarrow$ $\frac{dA}{dt} = 2\pi \times 4(1 - e^{-\lambda t}) \times 4\lambda e^{-\lambda t}$	M1, M1	
		$\frac{dA}{dt} = 32\pi\lambda(e^{-\lambda t} - e^{-2\lambda t})$	A1cso	5
	(b)	$\int A^{-\frac{3}{2}} dA = \int t^{-2} dt$	M1	
		Separation		
		$\frac{A^{-\frac{1}{2}}}{-\frac{1}{2}} = \frac{t^{-1}}{-1}$ (+ c)	M1, A1	
		$-2 = -1 + c$ Use of (1, 1)	M1	
		$c = -1$	A1	
		So $2A^{-\frac{1}{2}} = \frac{1}{t} + 1 \Rightarrow \sqrt{A} = \frac{2t}{1+t}$ Attempt $\sqrt{A}$ = or A =	M1	
		i.e. $A = \frac{4t^2}{(1+t)^2}$ (or equivalent)	A1	7
	(c)	Because $\frac{t^2}{(1+t)^2} < 1$ or $t^2 < (1 + t)^2$ ($\Rightarrow A < 4$)	B1	1

[13]

1. Many found part (a) difficult and it was quite common to see candidates leave a blank space here and proceed to solve part (b), often correctly. A satisfactory proof requires summarising the information given in the question in an equation, such as $\frac{dV}{dT} = 0.48\pi - 0.6\pi h$, but many could not do this or began with the incorrect $\frac{dh}{dt} = 0.48\pi - 0.6\pi h$. Some also found difficulty in obtaining a correct expression for the volume of water in the tank and there was some confusion as to which was the variable in expressions for the volume. Sometimes expressions of the form $V = \pi r^2 h$ were differentiated with respect to r, which in this question is a constant. If they started appropriately, nearly all candidates could use the chain rule correctly to complete the proof.

Part (b) was often well done and many fully correct solutions were seen. As noted in the introduction above, some poor algebra was seen in rearranging the equation but, if that was done correctly, candidates were nearly always able to demonstrate a complete method of solution although, as expected, slips were made in the sign and the constants when integrating. Very few candidates completed the question using definite integration. Most used a constant of integration (arbitrary constant) and showed that they knew how to evaluate it and use it to complete the question.

2. Connected rates of change is a topic which many find difficult. The examiners reported that the responses to this question were of a somewhat higher standard than had been seen in some recent examinations and the majority of candidates attempted to apply the chain rule to the data of the question. Among those who obtained a correct relation, $1.5 = 2\pi r \frac{dr}{dt}$ or an equivalent, a common error was to use $r = 2$, instead of using the given $A = 2$ to obtain $r = \sqrt{\frac{2}{\pi}}$.

Unexpectedly the use of the incorrect formula for the area of the circle, $A = 2\pi r^2$, was a relatively common error.

3. A considerable number of candidates did not attempt part (a), but of those who did, the most common method was to use similar triangles to obtain $r = \frac{2h}{3}$ and substitute r into $V = \frac{1}{3}\pi r^2 h$ to give $V = \frac{4}{27}\pi h^3$ Some candidates used trigonometry to find the semi-vertical angle of the cone and obtained $r = \frac{2h}{3}$ from this. A few candidates correctly used similar shapes to compare volumes by writing down the equation $\frac{V}{\frac{1}{3}\pi(16)^2 24} = \left(\frac{h}{24}\right)^3$.

Part (b) discriminated well between many candidates who were able to gain full marks with ease and some candidates who were able to gain just the first one or two marks. Some incorrectly differentiated $V = \frac{1}{3}\pi r^2 h$ to give $\frac{dV}{dh} = \frac{1}{3}\pi r^2$. Most of the successful candidates used the chain rule to find $\frac{dh}{dt}$ by applying $\frac{dV}{dt} \div \frac{dV}{dh}$. The final answer $\frac{1}{8\pi}$ was sometimes carelessly written as $\frac{1}{8}\pi$. Occasionally, some candidates solved the differential equation $\frac{dV}{dt} = 8$ and equated their solution to $\frac{4\pi h^3}{27}$ and then found $\frac{dt}{dh}$ or differentiated implicitly to find $\frac{dh}{dt}$.

4. At the outset, a significant minority of candidates struggled to extract some or all of the information from the question. These candidates were unable to write down the rate at which this cross-sectional area was increasing, $\frac{dA}{dt} = 0.032$; or the cross-sectional area of the cylinder $A = \pi x^2$ and its derivative $\frac{dA}{dt} = 2\pi x$; or the volume of the cylinder $V = 5\pi x^3$ and its derivative $\frac{dV}{dx} = 15\pi x^2$.

In part (a), some candidates wrote down the volume V of the cylinder as their cross-sectional area A. Another popular error at this stage was for candidates to find the curved surface area or the total surface area of a cylinder and write down either $A = 10\pi x^2$ or $A = 12\pi x^2$ respectively. At this stage many of these candidates were able to set up a correct equation to find $\frac{dx}{dt}$ and usually divided 0.032 by their $\frac{dA}{dx}$ and substituted $x = 2$ into their expression to gain 2 out of the 4 mark available. Another error frequently seen in part (a) was for candidates to incorrectly calculate $\frac{0.032}{4\pi}$ as 0.0251. Finally, rounding the answer to 3 significant figures proved to be a problem for a surprising number of candidates, with a value of 0.003 being seen quite often; resulting in loss of the final accuracy mark in part (a) and this sometimes as a consequence led to an inaccurate final answer in part (b).

Part (b) was tackled more successfully by candidates than part (a) – maybe because the chain rule equation $\frac{dV}{dt} = \frac{dV}{dx} + \frac{dx}{dt}$ is rather more straight-forward to use than the one in part (a). Some candidates struggled by introducing an extra variable r in addition to x and obtained a volume expression such as $V = \pi r^2(5x)$. Many of these candidates did not realise that $r \equiv x$ and were then unable to correctly differentiate their expression for V. Other candidates incorrectly wrote down the volume as $V = 2\pi x^2(5x)$. Another common error was for candidates to state a correct V, correctly find $\frac{dV}{dx}$, then substitute $x = 2$ to arrive at a final answer of approximately 188.5.

About 10% of candidates were able to produce a fully correct solution to this question.

5. The fact that this question had so many parts, with a good degree of independence, did enable the majority of candidates to do quite well. All but the weakest candidates scored the first mark and the first 3 were gained by most. The integration in part (c) did cause problems: examples of the more usual mistakes were to write

$\int \frac{1}{(2t+1)^2}\,dt = -\frac{1}{2t+1}$ or $\frac{1}{2t+1}$ or $\frac{2}{2t+1}$ or $\frac{k}{(2t+1)^3}$, or to omit the constant of integration or assume it equal to zero; two of the mistakes which came more into the "howler" category were

$\int \frac{1}{1000}\,dV = \ln 1000V$ *or* $V\ln 1000$ and

$\int \frac{1}{(2t+1)^2}dt = \int \frac{1}{4t^2+4t+1}\,dt = \int \frac{1}{4t^2} + \frac{1}{4t} + \frac{1}{1}dt =$ ....

Many candidates were able to gain the method marks in parts (d) and (e).

6. (a) Most understood the context of this problem and realised that they needed to use $t = 4$, although $t = 0$, 1 or 5 were often seen.

(b) Very few had any idea at all about how to differentiate V (many gave their answer as $-t(1.5)^{-t-1}$, or had a term $(1.5)^{-2t}$).

(c) The comments made in answer to the request to interpret their answer to part (b) were usually too generalised and vague. The examiners required a statement that the value of $\frac{dV}{dt}$ which had been found represented the rate of change of value on 1st January 2005

7. There were two common approaches used in part (a); substituting for r to obtain a formula for A in terms of t or using the chain rule. The inevitable errors involving signs and λ were seen with both methods and the examiners were disappointed that some candidates did not seem to know the formula for the area of a circle: $2\pi r, \frac{1}{2}\pi r^2$ and $4\pi r^2$ were common mistakes. Part (b) proved more testing. Most could separate the variables but the integration of negative powers caused problems for some who tried to use the ln function. Many did solve the differential equation successfully though sometimes they ran into difficulties by trying to make A the subject before finding the value of their arbitrary constant. The final two marks were only scored by the algebraically dexterous. There was some poor work here and seeing $\frac{2}{\sqrt{A}} = \frac{1}{t} + 1$ followed by $\frac{4}{A} = \frac{1}{t^2} + 1$ or $\frac{\sqrt{A}}{2} = t + 1$ was not uncommon. The final part eluded most. Those who had a correct answer to part (b) sometimes looked at the effect on A of $t \to \infty$ but only a small minority argued that since $t > 0$ then $t^2 < (1+t)^2$, and therefore $A < 4$.