Review exercise 2

1 a $x = 2\cot t, y = \sin^2 t$

$\frac{dx}{dt} = -2\operatorname{cosec}^2 t, \frac{dy}{dt} = 4\sin t \cos t$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= \frac{4\sin t \cos t}{-2\operatorname{cosec}^2 t}$

$= -2\sin^3 t \cos t$

b When $t = \frac{\pi}{4}$:

$x = 2$ and $y = 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 = 1$

$\frac{dy}{dx} = -2 \times \left(\frac{1}{\sqrt{2}}\right)^3 \times \left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

So equation of tangent is

$y - 1 = -\frac{1}{2}(x - 2)$

$y = -\frac{1}{2}x + 2$

c $x = 2\cot t \Rightarrow \cot t = \frac{x}{2}$

$y = 2\sin^2 t \Rightarrow \sin^2 t = \frac{y}{2}$ and $\operatorname{cosec}^2 t = \frac{2}{y}$

$\operatorname{cosec}^2 t = 1 + \cot^2 t$

$\frac{2}{y} = 1 + \left(\frac{x}{2}\right)^2$

$= \frac{4 + x^2}{4}$

$\frac{y}{2} = \frac{4}{4 + x^2}$

$y = \frac{8}{4 + x^2}$

As $0 < t \leqslant \frac{\pi}{2}, \cot t \geqslant 0$

Since $x = 2\cot t$, the domain of the function is $x \geqslant 0$.

2 a $x = \frac{1}{1+t}, y = \frac{1}{1-t}$

Using the chain rule:

$\frac{dx}{dt} = \frac{-1}{(1+t)^2}, \frac{dy}{dt} = \frac{1}{(1-t)^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= -\frac{(1+t)^2}{(1-t)^2}$

When $t = \frac{1}{2}$:

$x = \frac{2}{3}$ and $y = 2$

$\frac{dy}{dx} = -\frac{(\frac{3}{2})^2}{(\frac{1}{2})^2} = -\frac{\frac{9}{4}}{\frac{1}{4}} = -9$

So equation of tangent is

$y - 2 = -9\left(x - \frac{2}{3}\right)$

$y = -9x + 8$

b $x = \frac{1}{1+t} \Rightarrow t = \frac{1}{x} - 1$

Substitute into $y = \frac{1}{1-t}$:

$y = \frac{1}{1 - \left(\frac{1}{x} - 1\right)}$

$= \frac{1}{2 - \frac{1}{x}}$

$= \frac{x}{2x - 1}$

3 $3x^2 - 2y^2 + 2x - 3y + 5 = 0$

Differentiating with respect to x:

$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} + 0 = 0$

Substituting $x = 0,\ y = 1$:

$-4\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$

$7\frac{dy}{dx} = 2$

$\frac{dy}{dx} = \frac{2}{7}$

So gradient of normal at (0, 1) is $-\frac{7}{2}$

Equation of normal is

$y - 1 = -\frac{7}{2}(x - 0)$

$y = -\frac{7}{2}x + 1$

$7x + 2y - 2 = 0$

4 a $\sin x + \cos y = 0.5$

Differentiating with respect to x:

$\cos x - \sin y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{\cos x}{\sin y}$

b $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}$

When $x = \frac{\pi}{2}$:

$1 + \cos y = 0.5 \Rightarrow \cos y = -0.5$

$y = \frac{2\pi}{3}$ or $y = \frac{-2\pi}{3}$

When $x = -\frac{\pi}{2}$:

$-1 + \cos y = 0.5 \Rightarrow \cos y = 1.5$

(no solutions)

So the only stationary points in the given range are at $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{\pi}{2}, \frac{-2\pi}{3}\right)$.

5 a $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$

b Using the chain rule:

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dV}{dt} = \frac{1000}{(2t+1)^2} = 4\pi r^2 \times \frac{dr}{dt}$

So $\frac{dr}{dt} = \frac{1000}{4\pi(2t+1)^2 r^2}$

$= \frac{250}{\pi(2t+1)^2 r^2}$

6 a $x^3 + 3x^2 y = 4$

$3x^2 + 6xy + 3x^2\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{3x^2 + 6xy}{3x^2}$

$= -\frac{x + 2y}{x}$

b At the point (1, 1):

$\frac{dy}{dx} = -\frac{(1) + 2(1)}{(1)}$

$= -3$

Using $y - y_1 = m(x - x_1)$ with $m = -3$ at (1, 1) gives:

$y - 1 = -3(x - 1)$

$y - 1 = -3x + 3$

$y = -3x + 4$

7 a $x = \ln(2t - 1),\ y = at - 3t^3,\ t > k$

$2t - 1 > 0 \Rightarrow t > 0.5$

Therefore $k > 0.5$

7 b $\dfrac{dx}{dt}=\dfrac{2}{2t-1}$

$\dfrac{dy}{dt}=a-9t^2$

$\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}$

$=\dfrac{(2t-1)(a-9t^2)}{2}$

When $t=\dfrac{2}{3}$, $\dfrac{dy}{dx}=0$

$$\frac{\left(2\left(\frac{2}{3}\right)-1\right)\left(a-9\left(\frac{2}{3}\right)^2\right)}{2}=0$$

$$\frac{\frac{1}{3}(a-4)}{2}=0$$

$$a=4$$

c At $t=1$:

$x=\ln(2(1)-1)=0$

$y=4(1)-3(1)^3=1$

$\dfrac{dy}{dx}=\dfrac{(2(1)-1)(4-9(1)^2)}{2}$

$=-\dfrac{5}{2}$

Since the gradient of the tangent is $-\dfrac{5}{2}$,

the gradient of the normal is $\dfrac{2}{5}$

Using $y-y_1=m(x-x_1)$ with $m=\dfrac{2}{5}$ at (0, 1) gives:

$y-1=\dfrac{2}{5}(x-0)$

$y=\dfrac{2}{5}x+1$

8 a $x=\sec^2 t,\ y=\cot t,\ 0\leqslant t\leqslant\dfrac{\pi}{4}$

$x=\dfrac{1}{\cos^2 t}$

$\dfrac{dx}{dt}=\dfrac{0(\cos^2 x)-1(-\sin t\cos t+\cos t(-\sin x))}{\cos^4 t}$

$=\dfrac{2\sin t\cos t}{\cos^4 t}$

$=\dfrac{2\sin t}{\cos^3 t}$

$=\dfrac{2}{\cos^2 t}\left(\dfrac{\sin t}{\cos t}\right)$

$=2\sec^2 t\tan t$ as required

Therefore $k=2$

b $A=\displaystyle\int_0^{\frac{\pi}{4}} y\,dx$

From part **a**, $dx=2\sec^2 t\tan t\,dt$

$A=\displaystyle\int_0^{\frac{\pi}{4}}\cot t\left(2\sec^2 t\tan t\,dt\right)$

$=2\displaystyle\int_0^{\frac{\pi}{4}}\sec^2 t\,dt$

$=2[\tan x]_0^{\frac{\pi}{4}}$

$=2$

9 $x=\dfrac{2}{3}t^{\frac{3}{2}},\ y=2t^{\frac{5}{2}}$

$dx=t^{\frac{1}{2}}\,dt$

$A=\displaystyle\int_a^3 y\,dx=40$

$\displaystyle\int_a^3 2t^{\frac{5}{2}}\left(t^{\frac{1}{2}}\,dt\right)=40$

$\displaystyle\int_a^3 t^3\,dt=20$

$\left[\dfrac{1}{4}t^4\right]_a^3=20$

$3^4-a^4=80$

$a^4=3^4-80$

$=1$

$a=1$

10 a Crosses x-axis at $x = a$ so

$a > 0$ and $a\sqrt{1-a^2} = 0$

So $a = 1$

b $\pi\int_0^1 x^2(1-x^2)\,dx$

$= \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1$

$= \frac{2\pi}{15}$

11 $x = \tan t,\ y = \cos^2 t\ \ 0 \le t \le \frac{\pi}{4}$

$dx = \sec^2 t\,dt$

$V = \pi\int_0^{\frac{\pi}{4}} y^2\,dx$

$V = \pi\int_0^{\frac{\pi}{4}} \left(\cos^2 t\right)^2 \sec^2 t\,dt$

$= \pi\int_0^{\frac{\pi}{4}} \cos^2 t\,dt$

$= \frac{1}{2}\pi\int_0^{\frac{\pi}{4}} (1+\cos 2t)\,dt$

$= \frac{1}{2}\pi\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\pi\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$= \frac{\pi}{8}(\pi + 2)$

12 Let $I = \int_1^5 \frac{3x}{\sqrt{2x-1}}\,dx$

Let $u^2 = 2x - 1 \Rightarrow 2u\frac{du}{dx} = 2$

So replace dx with $u\,du$.

$\sqrt{2x-1} = u$ and $x = \frac{u^2+1}{2}$

x	u
1	1
5	3

So $I = \int_1^3 \frac{3}{2} \times \frac{u^2+1}{u} \times u\,du$

$= \int_1^3 \left(\frac{3}{2}u^2 + \frac{3}{2}\right)du$

$= \left[\frac{1}{2}u^3 + \frac{3}{2}u\right]_1^3$

$= \left(\frac{27}{2} + \frac{9}{2}\right) - \left(\frac{1}{2} + \frac{3}{2}\right)$

$= 18 - 2$

$= 16$

13 Let $I = \int_0^{\frac{1}{2}} \frac{x^3}{(1-x^2)^{\frac{1}{2}}} \, dx$

Let $u = 1 - x^2 \Rightarrow \frac{du}{dx} = -2x$

So replace $x \, dx$ with $-\frac{du}{2}$.

$x^2 = 1 - u$

$$\text{So } \int \frac{x^3}{(1-x^2)^{\frac{1}{2}}} dx = \int \frac{x^2}{(1-x^2)^{\frac{1}{2}}} x \, dx$$
$$= \int \frac{1-u}{u^{\frac{1}{2}}} \left(-\frac{du}{2}\right)$$
$$= -\frac{1}{2} \int \frac{1-u}{u^{\frac{1}{2}}} du$$
$$= -\frac{1}{2} \int \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du$$

x	u
$\frac{1}{2}$	$\frac{3}{4}$
0	1

$$\text{So } I = \left[-u^{\frac{1}{2}} + \frac{1}{3}u^{\frac{3}{2}}\right]_1^{\frac{3}{4}}$$
$$= \left(-\frac{\sqrt{3}}{2} + \frac{1}{3} \times \frac{3\sqrt{3}}{4\sqrt{4}}\right) - \left(-1 + \frac{1}{3}\right)$$
$$= \left(-\frac{3\sqrt{3}}{8}\right) - \left(-\frac{2}{3}\right)$$
$$= \frac{2}{3} - \frac{3\sqrt{3}}{8}$$

14 Let $I = \int_1^{e} (x^2 + 1) \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

and $\frac{dv}{dx} = x^2 + 1 \Rightarrow v = \frac{x^3}{3} + x$

Using the integration by parts formula:

$$I = \left[\left(\frac{x^3}{3} + x\right)\ln x\right]_1^{e} - \int_1^{e} \frac{1}{x}\left(\frac{x^3}{3} + x\right) dx$$
$$= \left(\frac{e^3}{3} + e\right) \times 1 - \left(\frac{1}{3} + 1\right) \times 0 - \int_1^{e} \left(\frac{x^2}{3} + 1\right) dx$$
$$= \frac{e^3}{3} + e - 0 - \left[\frac{x^3}{9} + x\right]_1^{e}$$
$$= \frac{e^3}{3} + e - \left(\left(\frac{e^3}{9} + e\right) - \left(\frac{1}{9} + 1\right)\right)$$
$$= \frac{2e^3}{9} + \frac{10}{9}$$
$$= \tfrac{1}{9}(2e^3 + 10)$$

15 a $$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{A}{2x-3} + \frac{B}{x+2}$$
$$\equiv \frac{A(x+2) + B(2x-3)}{(2x-3)(x+2)}$$
$$5x + 3 \equiv A(x+2) + B(2x-3)$$

Let $x = -2$: $-7 = B(-7)$ so $B = 1$

Let $x = \frac{3}{2}$: $\frac{21}{2} = A\left(\frac{7}{2}\right)$ so $A = 3$

So $$\frac{5x+3}{(2x-3)(x+2)} \equiv \frac{3}{2x-3} + \frac{1}{x+2}$$

15 b $\int_2^6 \frac{5x+3}{(2x-3)(x+2)}\,dx$

$= \int_2^6 \frac{3}{2x-3}\,dx + \int_2^6 \frac{1}{x+2}\,dx$

$= \left[\frac{3}{2}\ln(2x-3) + \ln(x+2)\right]_2^6$

$= \left(\frac{3}{2}\ln 9 + \ln 8\right) - \left(\frac{3}{2}\ln 1 + \ln 4\right)$

$= \ln 9^{\frac{3}{2}} + \ln 8 - 0 - \ln 4$

$= \ln 9^{\frac{3}{2}} + \ln \frac{8}{4}$

$= \ln 27 + \ln 2$

$= \ln 54$

16 $\int e^{-x}\cos 2x\,dx$

This is of the form

$\int u\,dv = uv - \int v\,du$

With $u = \cos 2x \Rightarrow \frac{du}{dx} = -2\sin 2x$ and

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$\int e^{-x}\cos 2x\,dx = -e^{-x}\cos 2x - \int -e^{-x}(-2\sin 2x)\,dx$

$\int e^{-x}\cos 2x\,dx = -e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx$ **(1)**

$\int e^{-x}\sin 2x\,dx$

is of the form

$\int u\,dv = uv - \int v\,du$

With $u = \sin 2x \Rightarrow \frac{du}{dx} = 2\cos 2x$ and

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

Substituting into $\int u\,dv = uv - \int v\,du$ gives:

$\int e^{-x}\sin 2x\,dx = -e^{-x}\sin 2x - \int -e^{-x}(2\cos 2x)\,dx$

$= -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx$

Substituting into **(1)** gives:

$\int e^{-x}\cos 2x\,dx$

$= -e^{-x}\cos 2x - 2\left(-e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx\right)$

$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 4\int e^{-x}\cos 2x\,dx$

$5\int e^{-x}\cos 2x\,dx$

$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x$

$\int e^{-x}\cos 2x\,dx$

$= \frac{2e^{-x}\sin 2x - e^{-x}\cos 2x}{5}$

$= \frac{e^{-x}}{5}(2\sin 2x - \cos 2x)$

17 a $\dfrac{2x-1}{(x-1)(2x-3)} \equiv \dfrac{A}{x-1} + \dfrac{B}{2x-3}$

$2x-1 \equiv A(2x-3) + B(x-1)$

Let $x = \frac{3}{2}$: $2 = B\left(\frac{1}{2}\right) \Rightarrow B = 4$

Let $x = 1$: $1 = A(-1) \Rightarrow A = -1$

So $\dfrac{2x-1}{(x-1)(2x-3)} \equiv \dfrac{-1}{x-1} + \dfrac{4}{2x-3}$

b $(2x-3)(x-1)\dfrac{dy}{dx} = (2x-1)y$

Separating the variables:

$\int \dfrac{1}{y}\,dy = \int \dfrac{2x-1}{(2x-3)(x-1)}\,dx$

So $\ln y = \int \dfrac{-1}{x-1}\,dx + \int \dfrac{4}{2x-3}\,dx$

$= -\ln|x-1| + 2\ln|2x-3| + c$

$= -\ln|x-1| + \ln(2x-3)^2 + \ln A$

$= \ln A\dfrac{(2x-3)^2}{x-1}$

So the general solution is

$y = \dfrac{A(2x-3)^2}{x-1}$

c $y = \dfrac{A(2x-3)^2}{x-1}$

When $x = 2$, $y = 10$ so

$10 = \dfrac{A(4-3)^2}{2-1} \Rightarrow A = 10$

So the particular solution is

$y = \dfrac{10(2x-3)^2}{(x-1)}$

18 a $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4\pi r^2$

Using the chain rule:

$\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$= 4\pi r^2 \times \dfrac{dr}{dt}$

So $\dfrac{k}{V} = 4\pi r^2 \times \dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{k}{\frac{4}{3}\pi r^3} \times \dfrac{1}{4\pi r^2}$

$= \dfrac{3k}{16\pi^2 r^5}$

So $B = \dfrac{3k}{16\pi^2}$

b Separating the variables:

$\int r^5\,dr = \int \dfrac{3k}{16\pi^2}\,dt$

$\dfrac{r^6}{6} = \dfrac{3k}{16\pi^2}t + A$

$r^6 = \dfrac{9k}{8\pi^2}t + A'$

$r = \left(\dfrac{9k}{8\pi^2}t + A'\right)^{\frac{1}{6}}$

19 a Rate of change of volume is $\dfrac{dV}{dt}$ cm^3 s^{-1}

Increase is 20 cm^3 s^{-1}

Decrease is kV cm^3 s^{-1}, where k is a constant of proportionality.

So the overall rate of change is

$\dfrac{dV}{dt} = 20 - kV$

19 b Separating the variables:

$$\int \frac{1}{20-kV}\,dV = \int 1\,dt$$

So $-\frac{1}{k}\ln|20-kV| = t + c$

When $t=0$, $V=0$ so

$$-\frac{1}{k}\ln 20 = c$$

Combining the ln terms:

$$-\frac{1}{k}\ln\frac{20-kV}{20} = t$$

$$\ln\frac{20-kV}{20} = -kt$$

$$\frac{20-kV}{20} = e^{-kt}$$

$$kV = 20 - 20e^{-kt}$$

$$V = \frac{20}{k} - \frac{20}{k}e^{-kt}$$

So $A = \frac{20}{k}$ and $B = -\frac{20}{k}$

c $V = \frac{20}{k} - \frac{20}{k}e^{-kt} \Rightarrow \frac{dV}{dt} = 20e^{-kt}$

Substitute $\frac{dV}{dt} = 10$ when $t = 5$:

$10 = 20e^{-5k} \Rightarrow e^{-5k} = \frac{1}{2}$

Taking natural logarithms:

$-5k = \ln\frac{1}{2}$ or $5k = \ln 2$

$k = \frac{1}{5}\ln 2 = 0.1386$ (4 d.p.)

So $V = \frac{100}{\ln 2} - \frac{100}{\ln 2}\left(\frac{1}{2}\right)^{\frac{t}{5}}$

When $t = 10$:

$$V = \frac{100}{\ln 2} - \frac{100}{\ln 2} \times \frac{1}{4}$$

$$= \frac{75}{\ln 2}$$

$$= 108.2 \text{ (1 d.p.)}$$

So the volume is 108 cm^3 (3 s.f.).

20 a $\frac{dC}{dt}$ is the rate of change of concentration.

The concentration is decreasing so the rate of change is negative.

So $-\frac{dC}{dt} \propto C$ or $\frac{dC}{dt} = -kC$,

where k is a positive constant of proportionality.

b Separating the variables:

$$\int \frac{1}{C}\,dC = -\int k\,dt$$

so $\ln C = -kt + \ln A$,

where $\ln A$ is a constant.

So $\ln\frac{C}{A} = -kt$

$$\frac{C}{A} = e^{-kt}$$

So the general solution is $C = Ae^{-kt}$.

c When $t = 0, C = C_0$ so $A = C_0$

So $C = C_0 e^{-kt}$

When $t = 4, C = \frac{1}{10}C_0$ so

$$\frac{1}{10}C_0 = C_0 e^{-4k}$$

$$e^{4k} = 10$$

$$4k = \ln 10$$

$$k = \frac{1}{4}\ln 10$$

21 The two vectors are parallel

so $9\mathbf{i} + q\mathbf{j} = \lambda(2\mathbf{i} - \mathbf{j})$

Equating coefficients:

$9 = 2\lambda$

$\lambda = \frac{9}{2}$

$q = -\lambda$

$= -\frac{9}{2}$

22 $|5\mathbf{i} - k\mathbf{j}| = |2k\mathbf{i} + 2\mathbf{j}|$

$\sqrt{5^2 + k^2} = \sqrt{(2k)^2 + 2}$

$25 + k^2 = 4k^2 + 4$

$3k^2 = 21$

$k^2 = 7$

$k = \pm\sqrt{7}$

The positive value of k is $\sqrt{7}$.

23 $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 8 \\ 23 \end{pmatrix} + \begin{pmatrix} -15 \\ x \end{pmatrix} = \begin{pmatrix} -7 \\ 23+x \end{pmatrix}$

or : $-7\mathbf{i} + (23 + x)\mathbf{j}$

$\mathbf{b} - \mathbf{c} = \begin{pmatrix} -15 \\ x \end{pmatrix} - \begin{pmatrix} -13 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ x-2 \end{pmatrix}$

or : $-2\mathbf{i} + (x - 2)\mathbf{j}$

As $\mathbf{a} + \mathbf{b}$ is parallel to $\mathbf{b} - \mathbf{c}$

$-7\mathbf{i} + (23 + x)\mathbf{j} = \lambda(-2\mathbf{i} + (x - 2)\mathbf{j})$

Equating coefficients and solving simultaneously:

$-7 = -2\lambda$ and $23 + x = \lambda(x - 2)$

$\lambda = 3.5$

$23 + x = 3.5(x - 2)$

$23 + x = 3.5x - 7$

$2.5x = 30$

$x = 12$

24 $|\overrightarrow{AB}| = \sqrt{1+36+16} = \sqrt{53}$

$|\overrightarrow{AC}| = \sqrt{25+4+9} = \sqrt{38}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = 6\mathbf{i} - 8\mathbf{j} - 7\mathbf{k}$

$|\overrightarrow{BC}| = \sqrt{36+64+49} = \sqrt{149}$

$\cos\angle BAC = \dfrac{53+38-149}{2\times\sqrt{53}\times\sqrt{38}} = -0.6462...$

$\angle BAC = 130.3°$ (1 d.p.)

25 a Let O be the fixed origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 10\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{100+25+4} = \sqrt{129}$

Unit vector in direction of $\overrightarrow{PQ}$

$= \dfrac{10}{\sqrt{129}}\mathbf{i} - \dfrac{5}{\sqrt{129}}\mathbf{j} - \dfrac{2}{\sqrt{129}}\mathbf{k}$

c $\cos\theta_z = \dfrac{-2}{\sqrt{129}} = -0.1761$

$\theta_z = 101.1°$ (1 d.p.)

d $\overrightarrow{AB} = 30\mathbf{i} - 15\mathbf{j} + 6\mathbf{k}$

There is no scalar, say m, for which $\overrightarrow{AB} = m\overrightarrow{PQ}$, so $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ are not parallel.

26 $\overrightarrow{MN} = 10\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$

$|\overrightarrow{MN}| = \sqrt{10^2 + 5^2 + 4^2} = \sqrt{141}$

$\overrightarrow{MP} = (k+2)\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$

$|\overrightarrow{MP}| = \sqrt{(k+2)^2 + 2^2 + 11^2}$

$= \sqrt{(k+2)^2 + 125}$

$\overrightarrow{NP} = (k-8)\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$

$|\overrightarrow{NP}| = \sqrt{(k-8)^2 + 3^2 + 7^2}$

$= \sqrt{(k-8)^2 + 56}$

If $|\overrightarrow{MN}| = |\overrightarrow{MP}|$ then

$\sqrt{141} = \sqrt{(k+2)^2 + 125}$

$(k+2)^2 = 16$

$k + 2 = \pm 4$

$k = 2$ or $k = -6$

$\Rightarrow k = 2$ (since k is positive)

If $|\overrightarrow{MN}| = |\overrightarrow{NP}|$ then

$\sqrt{141} = \sqrt{(k-8)^2 + 56}$

$(k-8)^2 = 85$

So there are no integer solutions for k if $|\overrightarrow{MN}| = |\overrightarrow{NP}|$

If $|\overrightarrow{MP}| = |\overrightarrow{NP}|$ then

$\sqrt{(k+2)^2 + 125} = \sqrt{(k-8)^2 + 56}$

$k^2 + 4k + 129 = k^2 - 16k + 122$

$20k = -7$

So there are no positive solutions for k if $|\overrightarrow{MP}| = |\overrightarrow{NP}|$

So $k = 2$

27 $-6\mathbf{i} + 40\mathbf{j} + 16\mathbf{k} = 3p\mathbf{i} + (8 + qr)\mathbf{j} + 2pr\mathbf{k}$

Comparing coefficients of **i**:
$-6 = 3p \Rightarrow p = -2$

Comparing coefficients of **k**:
$16 = 2pr \Rightarrow pr = 8 \Rightarrow r = -4$

Comparing coefficients of **j**:
$40 = 8 + qr \Rightarrow qr = 32 \Rightarrow q = -8$

$p = -2, q = -8, r = -4$

28 $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

At A, $\lambda = 4$ and at B, $\lambda = -1$

For A:

$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + 4\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -9 \\ 15 \end{pmatrix}$

So A is the point (−2, −9, 15)

For B:

$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - 1\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

So B is the point (3, 1, 0)

$|AB| = \sqrt{(-2-3)^2 + (-9-1)^2 + (15-0)^2}$

$= \sqrt{(-5)^2 + (-10)^2 + (15-0)^2}$

$= 5\sqrt{14}$

29 P is the point (1, −1, 3), Q is the point (a, 3, 8) and R is the point (5, 7, b)

$$\overrightarrow{PQ}=\begin{pmatrix}a\\3\\8\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}a-1\\4\\5\end{pmatrix}$$

$$\overrightarrow{PR}=\begin{pmatrix}5\\7\\b\end{pmatrix}-\begin{pmatrix}1\\-1\\3\end{pmatrix}=\begin{pmatrix}4\\8\\b-3\end{pmatrix}$$

$$\overrightarrow{QR}=\begin{pmatrix}5\\7\\b\end{pmatrix}-\begin{pmatrix}a\\3\\8\end{pmatrix}=\begin{pmatrix}5-a\\4\\b-8\end{pmatrix}$$

Since the points are collinear:

$$\begin{pmatrix}a-1\\4\\5\end{pmatrix}=\lambda\begin{pmatrix}4\\8\\b-3\end{pmatrix}=\mu\begin{pmatrix}5-a\\4\\b-8\end{pmatrix}$$

From the second row:

$8\lambda=4$

$\lambda=\frac{1}{2}$

From first row:

$a-1=4\left(\frac{1}{2}\right)$

$a=3$

From third row:

$\frac{1}{2}(b-3)=5$

$b=13$

Substituting for a and b gives:

$$\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\4\\5\end{pmatrix}$$

30 a l_1 has equation $\mathbf{r}=\begin{pmatrix}11\\5\\6\end{pmatrix}+\lambda\begin{pmatrix}4\\2\\4\end{pmatrix}$ and l_2 has equation $\mathbf{r}=\begin{pmatrix}24\\4\\13\end{pmatrix}+\mu\begin{pmatrix}7\\1\\5\end{pmatrix}$

If l_1 and l_2 intersect then:

$$\begin{pmatrix}11\\5\\6\end{pmatrix}+\lambda\begin{pmatrix}4\\2\\4\end{pmatrix}=\begin{pmatrix}24\\4\\13\end{pmatrix}+\mu\begin{pmatrix}7\\1\\5\end{pmatrix}$$

$11+4\lambda=24+7\mu\Rightarrow 4\lambda-7\mu=13$ **(1)**

$5+2\lambda=4+\mu\Rightarrow 2\lambda-\mu=-1$ **(2)**

$6+4\lambda=13+5\mu\Rightarrow 4\lambda-5\mu=7$ **(3)**

Subtracting **(3)** from **(1)** gives:

$4\lambda-7\mu-4\lambda+5\mu=13-7$

$-2\mu=6$

$\mu=-3$

Substituting $\mu=-3$ into **(3)** gives:

$4\lambda-5(-3)=7$

$4\lambda=-8$

$\lambda=-2$

Substituting $\mu=-3$ and $\lambda=-2$ into **(2)** gives:

$\text{LHS}=2(-2)-(-3)=-1$

$\text{RHS}=-1$

$-1=-1$

Therefore l_1 and l_2 intersect

b Substituting $\lambda=-2$ into $\begin{pmatrix}11\\5\\6\end{pmatrix}+\lambda\begin{pmatrix}4\\2\\4\end{pmatrix}$ gives:

$$\begin{pmatrix}11\\5\\6\end{pmatrix}-2\begin{pmatrix}4\\2\\4\end{pmatrix}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$$

So l_1 and l_2 intersect at the point (3, 1, −2)

30 c $\cos\theta = \dfrac{l_1 \cdot l_2}{|l_1||l_2|}$

$l_1 \cdot l_2 = \begin{pmatrix}4\\2\\4\end{pmatrix} \cdot \begin{pmatrix}7\\1\\5\end{pmatrix}$

$= 4(7) + 2(1) + 4(5)$

$= 50$

$|l_1| = \sqrt{4^2 + 2^2 + 4^2}$

$= \sqrt{36}$

$= 6$

$|l_2| = \sqrt{7^2 + 1^2 + 5^2}$

$= \sqrt{75}$

$\cos\theta = \dfrac{50}{6\sqrt{75}}$

$= \dfrac{5\sqrt{3}}{9}$

31 a l_1 has equation $\mathbf{r} = \begin{pmatrix}8\\12\\14\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$A(4, 8, a)$ and $B(b, 13, 13)$ lie on l_1

For A:

$\begin{pmatrix}8\\12\\14\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}4\\8\\a\end{pmatrix}$

$8 + \lambda = 4 \Rightarrow \lambda = -4$

$14 - \lambda = a \Rightarrow a = 18$

For B:

$\begin{pmatrix}8\\12\\14\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}b\\13\\13\end{pmatrix}$

$12 + \lambda = 13 \Rightarrow \lambda = 1$

$8 + \lambda = b \Rightarrow b = 9$

So $a = 18$ and $b = 9$

31 b Let P be the point (x, y, z)

l_1 has equation $\mathbf{r} = \begin{pmatrix}8\\12\\14\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$\overrightarrow{OP}$ is perpendicular to l_1, therefore:

$\begin{pmatrix}1\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = 0$

$x + y - z = 0$ **(1)**

Since P lies on l_1:

$\begin{pmatrix}8\\12\\14\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

$x = 8 + \lambda$

$y = 12 + \lambda$

$z = 14 - \lambda$

Substituting for x, y and z in **(1)** gives:

$(8 + \lambda) + (12 + \lambda) - (14 - \lambda) = 0$

$3\lambda = -6$

$\lambda = -2$

When $\lambda = -2$

$x = 8 + (-2) = 6$

$y = 12 + (-2) = 10$

$z = 14 - (-2) = 16$

So P has coordinates (6, 10, 16)

c $|\overrightarrow{OP}| = \sqrt{6^2 + 10^2 + 16^2}$

$= 14\sqrt{2}$

32 a The shark has position vector $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and swims to $\begin{pmatrix}-2\\11\\11\end{pmatrix}$

The flounder has position vector $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and swims in the direction $\begin{pmatrix}-2\\-1\\3\end{pmatrix}$

For the shark:

$$\begin{pmatrix}-2\\11\\11\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}-4\\8\\10\end{pmatrix}$$

$$\mathbf{r}=\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\8\\10\end{pmatrix}$$

For the flounder:

$$\mathbf{r}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}$$

If the paths of the shark and flounder intersect then:

$$\begin{pmatrix}2\\3\\-1\end{pmatrix}+\lambda\begin{pmatrix}-4\\8\\10\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}$$

$2-4\lambda=2-2\mu \Rightarrow -4\lambda+2\mu=0$ **(1)**

$3+8\lambda=-\mu \Rightarrow 8\lambda+\mu=-3$ **(2)**

$-1+10\lambda=1+3\mu \Rightarrow 10\lambda-3\mu=2$ **(3)**

Adding 2 × **(1)** and **(2)** gives:

$-8\lambda+4\mu+8\lambda+\mu=0-3$

$5\mu=-3$

$\mu=-\frac{3}{5}$

Substituting $\mu=-\frac{3}{5}$ into **(1)** gives:

$-4\lambda+2\left(-\frac{3}{5}\right)=0$

$\lambda=-\frac{3}{10}$

$\lambda=-\frac{3}{10}$

Substituting $\lambda=-\frac{3}{10}$ and $\mu=-\frac{3}{5}$ into **(3)** gives:

$\text{LHS}=10\left(-\frac{3}{10}\right)-3\left(-\frac{3}{5}\right)=-\frac{6}{5}$

$\text{RHS}=2$

$-\frac{6}{5}\neq 2$

Therefore the paths of the shark and flounder do not intersect.

b It is unlikely that the shark will not adjust course to intercept the flounder.
Fish don't tend to swim in straight lines.

Challenge

1 a $x^3 - xy^2 = y + 5$

$$3x^2 - y^2 - 2xy\frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx}(1+2xy) = 3x^2 - y^2$$

$$\frac{dy}{dx} = \frac{3x^2 - y^2}{1+2xy}$$

b $\frac{dy}{dx}(1+2xy) = 3x^2 - y^2$

$$\frac{d^2y}{dx^2}(1+2xy) + \frac{dy}{dx}\left(2y + 2x\frac{dy}{dx}\right) = 6x - 2y\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}(1+2xy) + 2y\frac{dy}{dx} + 2x\left(\frac{dy}{dx}\right)^2 = 6x - 2y\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}(1+2xy) = 6x - 4y\frac{dy}{dx} - 2x\left(\frac{dy}{dx}\right)^2$$

$$\frac{d^2y}{dx^2} = \frac{6x - 4y\frac{dy}{dx} - 2x\left(\frac{dy}{dx}\right)^2}{1+2xy}$$

c At (2, 1):

$$\frac{dy}{dx} = \frac{3(2)^2 - (1)^2}{1+2(2)(1)}$$

$$= \frac{11}{5}$$

$$\frac{d^2y}{dx^2} = \frac{6(2) - 4(1)\left(\frac{11}{5}\right) - 2(2)\left(\frac{11}{5}\right)^2}{1+2(2)(1)}$$

$$= -\frac{404}{125}$$

2 $\int_{e^2}^{e^3} \frac{1}{x\ln x}\,dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

The limits become:

$u = \ln e^2 = 2$ and $u = \ln e^3 = 3$

$$\int_{e^2}^{e^3} \frac{1}{x\ln x}\,dx = \int_2^3 \frac{1}{u}\,du$$

$$= [\ln u]_2^3$$

$$= \ln 3 - \ln 2$$

$$= \ln\left(\frac{3}{2}\right)$$

3 $\overrightarrow{OA} = \begin{pmatrix}-2\\-3\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}-1\\-1\\3\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$\overrightarrow{AB} = \begin{pmatrix}-1\\-1\\3\end{pmatrix} - \begin{pmatrix}-2\\-3\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\2\\-2\end{pmatrix}$$

$$\overrightarrow{AB}\cdot\overrightarrow{BC} = \begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\2\\-2\end{pmatrix}$$

$$= 1(2) + 2(2) + 3(-2)$$

$$= 0$$

Therefore $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular and hence $\overrightarrow{AC}$ is a diameter.

The midpoint of $\overrightarrow{AC}$ is:

$$\left(\frac{-2+1}{2}, \frac{-3+1}{2}, \frac{0+1}{2}\right) = \left(-\frac{1}{2}, -1, \frac{1}{2}\right)$$

$$\left|\overrightarrow{AC}\right| = \sqrt{(-2-1)^2 + (-3-1)^2 + (0-1)^2}$$

$$= \sqrt{26}$$

Therefore the radius is $\frac{\sqrt{26}}{2}$ and the midpoint is $\left(-\frac{1}{2}, -1, \frac{1}{2}\right)$