Review exercise 1

1 Assumption: there are a finite number of prime numbers, p_1, p_2, p_3 up to p_n.

Let $X = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$

None of the prime numbers $p_1, p_2, p_3 \ldots p_n$ can be a factor of X as they all leave a remainder of 1 when X is divided by them. But X must have at least one prime factor. This is a contradiction. So there are infinitely many prime numbers.

2 Assumption: $x = \frac{a}{b}$ is a solution to the equation,

$x^2 - 2 = 0$, where a and b are integers with no common factors.

$$\left(\frac{a}{b}\right)^2 - 2 = 0 \Rightarrow \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$$

So a^2 is even, which implies that a is even.
Write $a = 2n$ for some integer n.

$$(2n)^2 = 2b^2 \Rightarrow 4n^2 = 2b^2 \Rightarrow 2n^2 = b^2$$

So b^2 is even, which implies that b is even.
This contradicts the assumption that a and b have no common factor. Hence there are no rational solutions to the equation.

3 Assumption: if n is odd then $3n^2 + 2$ is even.
Let $n = 2k + 1$ where k is an integer.
Then $3(2k + 1)^2 + 2 = 3(4k^2 + 4k + 1) + 2 = 12(k^2 + k) + 5$
This is an even number plus an odd number which must be odd.
This contradicts the assumption made.
Therefore if n is odd then $3n^2 + 2$ is odd.

4 Assumption: $\sqrt{5}$ is rational.

Let $\sqrt{5} = \frac{a}{b}$ for integers a and b.

Also assume that this fraction is in its simplified form and there are no common factors.

Then $5 = \frac{a^2}{b^2}$ or $a^2 = 5b^2$

Hence a^2 must be a multiple of 5.
Since a is an integer it follows that a is also a multiple of 5.
So let $a = 5c$ where c is an integer.
Then $a^2 = 25c^2$, and so $5b^2 = 25c^2$ which leads to $b^2 = 5c^2$
Now b^2 is a multiple of 5, and so b is a multiple of 5.
If a and b are both multiples of 5 then this contradicts the initial statement of there being no common factors.
Hence $\sqrt{5}$ is irrational.

5 $\frac{2x-1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$

$\Rightarrow 2x-1 = A(2x-3) + B(x-1)$

Set $x = 1$: $2(1)-1 = 1 = A(2(1)-3) = -A$
$\Rightarrow A = -1$

Set $x = \frac{3}{2}$: $2\left(\frac{3}{2}\right)-1 = 2 = B\left(\frac{3}{2}-1\right) = \frac{1}{2}B$
$\Rightarrow B = 4$

So $\frac{2x-1}{(x-1)(2x-3)} = \frac{-1}{x-1} + \frac{4}{2x-3}$

6 $\frac{3x+7}{(x+1)(x+2)(x+3)} = \frac{P}{x+1} + \frac{Q}{x+2} + \frac{R}{x+3}$

$\Rightarrow 3x+7 = P(x+2)(x+3)Q(x+1)(x+3) + R(x+1)(x+2)$

Set $x = -1$: $3(-1)+7 = 4 = P((-1)+2)((-1)+3) = 2P$
$\Rightarrow P = 2$

Set $x = -2$: $3(-2)+7 = 1 = Q((-2)+1)((-2)+3) = -Q$
$\Rightarrow Q = -1$

Set $x = -3$: $3(-3)+7 = -2 = R((-3)+1)((-3)+2) = 2R$
$\Rightarrow R = -1$

So $P = 2, Q = -1, R = -1$

7 $\frac{2}{(2-x)(1+x)^2} = \frac{A}{2-x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$

$\Rightarrow 2 = A(1+x)^2 + B(1+x)(2-x) + C(2-x)$
Set $x = 2$: $2 = A(1+2)^2 = 9A$ so $A = \frac{2}{9}$
Set $x = -1$: $2 = C[2-(-1)] = 3C$ so $C = \frac{2}{3}$
Compare coefficients of x^2: $0 = A - B$
$\Rightarrow B = A = \frac{2}{9}$
Solution: $A = \frac{2}{9}, B = \frac{2}{9}, C = \frac{2}{3}$

8 $\frac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{2x+1} + \frac{C}{(2x+1)^2}$

$\equiv \frac{A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)}{(x+1)(2x+1)^2}$

Compare numerators of fractions:

$14x^2+13x+2 = A(2x+1)^2 + B(x+1)(2x+1) + C(x+1)$

Put $x = -1$

$3 = A + 0 + 0 \Rightarrow A = 3$

Put $x = -\frac{1}{2}$

$\frac{14}{4} - \frac{13}{2} + 2 = \frac{1}{2}C \Rightarrow C = -2$

So $14x^2+13x+2 = 3(2x+1)^2 + B(x+1)(2x+1) - 2(x+1)$

Compare coefficients of x^2:

$14 = 12 + 2B \Rightarrow B = 1$

Check constant term:

$2 = 3 + 1 - 2$

So $\frac{14x^2+13x+2}{(x+1)(2x+1)^2} \equiv \frac{3}{x+1} + \frac{1}{2x+1} - \frac{2}{(2x+1)^2}$

9 $\frac{3x^2+6x-2}{x^2+4} \equiv d + \frac{ex+f}{x^2+4}$

$\Rightarrow 3x^2+6x-2 = d(x^2+4) + ex + f$

Compare coefficients of x^2 : $3 = d$
Compare coefficients of x : $6 = e$
Compare constant terms: $-2 = 4d + f$

So $f = -2 - 4d = -2 - 4(3) = -14$

Solution: $d = 3$, $e = 6$, $f = -14$

10 $\mathrm{p}(x) = \frac{9-3x-12x^2}{(1-x)(1+2x)} = A + \frac{B}{1-x} + \frac{C}{1+2x}$

$\Rightarrow 9-3x-12x^2 = A(1-x)(1+2x) + B(1+2x) + C(1-x)$

Set $x = 1$: $9 - 3(1) - 12(1)^2 = -6 = B(1 + 2(1)) = 3B$

$\Rightarrow B = -2$

Set $x = -\frac{1}{2}$: $9 - 3\left(-\frac{1}{2}\right) - 12\left(-\frac{1}{2}\right)^2 = \frac{15}{2} = C\left(1 - \left(-\frac{1}{2}\right)\right) = \frac{3}{2}C$

$\Rightarrow C = 5$

Compare coefficients of x^2 : $-12 = -2A$

$\Rightarrow A = 6$

Solution: $A = 6, B = 1, C = -2$

11 $\dfrac{4x-1}{(x+1)(x+3)}=\dfrac{A}{x+1}+\dfrac{B}{x+3}$

$4x-1=A(x+3)+B(x+1)$

$\quad=Ax+3A+Bx+B$

$\quad=(A+B)x+(3A+B)$

Comparing coefficients

For x:

$A+B=4$ **(1)**

$3A+B=-1$ **(2)**

Subtracting **(1)** from **(2)** gives:

$3A+B-A-B=-1-4$

$2A=-5$

$A=-\dfrac{5}{2}$

Substituting $A=-\dfrac{5}{2}$ into **(1)** gives:

$\left(-\dfrac{5}{2}\right)+B=4$

$B=\dfrac{13}{2}$

Therefore:

$\dfrac{4x-1}{(x+1)(x+3)}=-\dfrac{5}{2(x+1)}+\dfrac{13}{2(x+3)}$

12 $\dfrac{4x^3}{(x-3)(x-1)^2} = A = \dfrac{B}{x-3} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

$\dfrac{4x^3}{(x-3)(x-1)^2} = A(x-3)(x-1)^2 + B(x-1)^2 + C(x-3)(x-1) + D(x-3)$

$= Ax^3 - 5Ax^2 + 7Ax - 3A + Bx^2 - 2Bx + B + Cx^2 - 4Cx + 3C + Dx - 3D$

$= Ax^3 + (B - 5A + C)x^2 + (7A - 2B - 4C + D)x + (B - 3A + 3C - 3D)$

Comparing coefficients

For x^3:

$A = 4$

For x^2:

$B - 5A + C = 0 \Rightarrow B + C = 20$ **(1)**

For x:

$7A - 2B - 4C + D = 0 \Rightarrow -2B - 4C + D = -28$ **(2)**

For constant:

$B - 3A + 3C - 3D = 0 \Rightarrow B + 3C - 3D = 12$ **(3)**

Adding 3 × **(2)** to **(3)** gives:

$B + 3C - 3D - 6B - 12C + 3D = 12 - 84$

$5B + 9C = 72$ **(4)**

Subtracting 5 × **(1)** from **(4)** gives:

$5B + 9C - 5B - 5C = 72 - 100$

$4C = -28$

$C = -7$

Substituting $C = -7$ into **(4)** gives:

$5B + 9(-7) = 72$

$B = 27$

Substituting $B = 27$ and $C = -7$ into **(2)** gives:

$-2(27) - 4(-7) + D = -28$

$D = -2$

Therefore:

$A = 4$, $B = 27$, $C = -7$ and $D = -2$

13 a $\dfrac{5x+3}{(2x-3)(x+2)}=\dfrac{A}{2x-3}+\dfrac{B}{x+2}$

$5x+3=A(x+2)+B(2x-3)$

$=Ax+2A+2Bx-3B$

$=(A+2B)x+(2A-3B)$

Comparing coefficients:
For x:
$A+2B=5$ **(1)**
For constant:
$2A-3B=3$ **(2)**
Subtracting 2 × **(1)** from **(2)** gives:
$2A-3B-2A-4B=3-10$
$-7B=-7$
$B=1$
Substituting $B=1$ into **(1)** gives:
$A+2(1)=5$
$A=3$
Therefore:

$\dfrac{5x+3}{(2x-3)(x+2)}=\dfrac{3}{2x-3}+\dfrac{1}{x+2}$

b $\displaystyle\int_2^6\frac{5x+3}{(2x-3)(x+2)}\,dx=\int_2^6\left(\frac{3}{2x-3}+\frac{1}{x+2}\right)dx$

$=\left[\dfrac{3}{2}\ln|2x-3|+\ln|x+2|\right]_2^6$

$=\left(\dfrac{3}{2}\ln|2(6)-3|+\ln|(6)+2|\right)-\left(\dfrac{3}{2}\ln|2(2)-3|+\ln|(2)+2|\right)$

$=\left(\dfrac{3}{2}\ln 9+\ln 8\right)-\left(\dfrac{3}{2}\ln 1+\ln 4\right)$

$=\dfrac{3}{2}\ln 9+\ln 2$

$=\ln 27+\ln 2$

$=\ln 54$

14 a As $\frac{4}{t} \neq 0, x \neq 1$

The equation for y can be rewritten as

$y = \left(t - \frac{3}{2}\right)^2 - \frac{5}{4}$

So $y \geqslant -\frac{5}{4}$

b $t = \frac{4}{1-x}$

$$\text{So } y = \left(\frac{4}{1-x}\right)^2 - 3\left(\frac{4}{1-x}\right) + 1$$

$$= \frac{16}{(1-x)^2} - \frac{12(1-x)}{(1-x)^2} + \frac{(1-x)^2}{(1-x)^2}$$

$$= \frac{16 - 12 + 12x + 1 - 2x + x^2}{(1-x)^2}$$

$$= \frac{x^2 + 10x + 5}{(1-x)^2}$$

So $a = 1,\ b = 10,\ c = 5$

15 a $x = \ln(t+2) \Rightarrow e^x = t + 2 \Rightarrow t = e^x - 2$

$$y = \frac{3t}{t+3} = \frac{3e^x - 6}{e^x + 1}$$

$t > 4 \Rightarrow e^x - 2 > 4 \Rightarrow e^x > 6 \Rightarrow x > \ln 6$

So the solution is $y = \frac{3e^x - 6}{e^x + 1}, x > \ln 6$

b When $x \to \infty,\ y \to 3$

When $x = \ln 6$, $y = \frac{3e^{\ln 6} - 6}{e^{\ln 6} + 1} = \frac{(3 \times 6) - 6}{6 + 1} = \frac{12}{7}$

So range is $\frac{12}{7} < y < 3$

16 $x = \frac{1}{1+t} \Rightarrow t = \frac{1}{x} - 1 = \frac{1-x}{x}$

$$y = \frac{1}{1 - \left(\frac{1-x}{x}\right)} = \frac{x}{x - (1-x)} = \frac{x}{2x - 1}$$

17 a $y = \cos 3t = \cos(2t + t) = \cos 2t \cos t - \sin 2t \sin t$

$= (\cos^2 t - 1)\cos t - 2\sin^2 t \cos t$

$= 2\cos^3 t - \cos t - 2(1 - \cos^2 t)\cos t$

$= 4\cos^3 t - 3\cos t$

$x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$

$y = 4\left(\frac{x}{2}\right)^3 - 3\left(\frac{x}{2}\right) = \frac{1}{2}x^3 - \frac{3}{2}x = \frac{x}{2}(x^2 - 3)$

b $0 \leqslant t \leqslant \frac{\pi}{2}$

So $0 \leqslant \cos t \leqslant 1$ and $-1 \leqslant \cos 3t \leqslant 1$

So $0 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 1$

18 a $y = \sin\left(t + \frac{\pi}{6}\right) = \sin t \cos\frac{\pi}{6} + \cos t \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\cos t$

$= \frac{\sqrt{3}}{2}\sin t + \frac{1}{2}\sqrt{1 - \sin^2 t}$

$= \frac{\sqrt{3}}{2}x + \frac{1}{2}\sqrt{1 - x^2}$

Note that we have to take the positive square root $\sqrt{1 - \sin^2 t}$ since $\cos t \geqslant 0$ for $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$

As $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}, -1 \leqslant \sin t \leqslant 1 \Rightarrow -1 \leqslant x \leqslant 1$

b At A, $\sin\left(t + \frac{\pi}{6}\right) = 0 \Rightarrow t = -\frac{\pi}{6}$

$x = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Coordinates of A are $\left(-\frac{1}{2}, 0\right)$

At B, $x = \sin t = 0 \Rightarrow t = 0$

$y = \sin\left(t + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

Coordinates of B are $\left(0, \frac{1}{2}\right)$

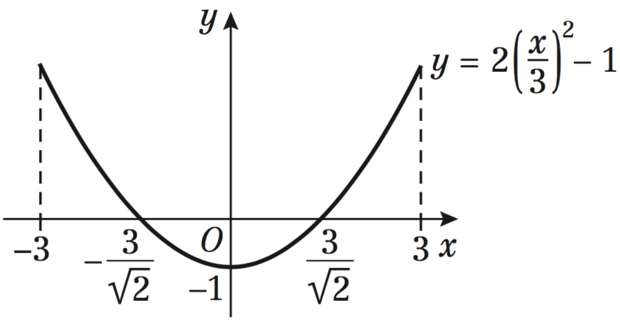
19 a $y = \cos 2t = 2\cos^2 t - 1$

$y = 2\left(\frac{x}{3}\right)^2 - 1, \; -3 \leqslant x \leqslant 3$

b Curve is a parabola, with a minima and y-intercept at (0,–1) and x-intercepts when

$2\left(\frac{x}{3}\right)^2 = 1 \Rightarrow \frac{x}{3} = \pm\frac{1}{\sqrt{2}} \Rightarrow x = \pm\frac{3}{\sqrt{2}}$

Coordinates $\left(-\frac{3}{\sqrt{2}}, 0\right), \left(\frac{3}{\sqrt{2}}, 0\right)$



20 a $(1-2x)^{10} = 1 + (10)(-2x) + \frac{(10)(9)}{2!}(-2x)^2 + \frac{(10)(9)(8)}{3!}(-2x)^3$

$= 1 - 20x + 180x^2 - 960x^3$

b $(0.98)^{10} = (1-0.02)^{10}$

Using $(1-2x)^{10} = 1 - 20x + 180x^2 - 960x^3$ with $x = 0.01$ gives:

$(0.98)^{10} = 1 - 20(0.01) + 180(0.01)^2 - 960(0.01)^3$

$= 0.81704$

$= 0.817$ (3 d.p.)

21 $(2-x)(1+2x)^5 = (2-x)\left(1 + (5)(2x) + \frac{(5)(4)}{2!}(2x)^2 + \ldots\right)$

$= (2-x)(1 + 10x + 40x^2 + \ldots)$

$= 2 + 20x + 80x^2 - x - 10x^2 - \ldots$

$= 2 + 19x + 70x^2$

So $a = 2$, $b = 19$ and $c = 70$

22 $(2-4x)^q = 2^q(1-2x)^q$

The x term is:

$2^q(q)(-2x)$

Therefore the coefficient of x is given by:

$(2^q)(-2q)$

Since the coefficient of $x = -32q$

$(2^q)(-2q) = -32q$

$$2^q = 16$$
$$q = 4$$

23 a Using the binomial expansion

$$g(x) = (1-x)^{-\frac{1}{2}}$$
$$= 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x)^3 + \ldots$$
$$= 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \ldots$$

b $|x| < 1$

24 a $(1+ax)^n \equiv 1 + nax + \dfrac{n(n-1)}{2}a^2x^2 + \ldots$

$na = -6$ **(1)**

$\dfrac{n(n-1)}{2}a^2 = 45$ **(2)**

From equation **(1)** $a = -\dfrac{6}{n}$

Substitute into equation **(2)**

$$\frac{n(n-1)}{2} \times \frac{36}{n^2} = 45$$
$$36n^2 - 36n = 90n^2$$
$$-36n = 54n^2$$

$\Rightarrow n = 0$ or $n = -\frac{36}{54} = -\frac{2}{3}$

Substitute into equation **(1)** to give $a = 9$

b Coefficient of $x^3 = \dfrac{n(n-1)(n-2)}{3!}a^3$

$$= \frac{-\frac{2}{3} \times -\frac{5}{3} \times -\frac{8}{3} \times 9^3}{3!}$$
$$= \frac{-80 \times 27}{6}$$
$$= -360$$

24 c The expansion is valid if $|9x| < 1$

So $-\frac{1}{9} < x < \frac{1}{9}$

25 a Using the binomial expansion

$$(1+4x)^{\frac{3}{2}} = 1 + \left(\tfrac{3}{2}\right)(4x) + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(4x)^2 + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{3!}(4x)^3 + \ldots$$

$$= 1 + 6x + 6x^2 - 4x^3 + \ldots$$

b $$\left(1+4\left(\frac{3}{100}\right)\right)^{\frac{3}{2}} = \left(\frac{112}{100}\right)^{\frac{3}{2}} = \left(\sqrt{\frac{112}{100}}\right)^3 = \frac{112\sqrt{122}}{1000}$$

c $$1 + 6\left(\frac{3}{100}\right) + 6\left(\frac{3}{100}\right)^2 - 4\left(\frac{3}{100}\right)^3 = 1.185292$$

So $\dfrac{112\sqrt{112}}{1000} \approx 1.185292$

$$\Rightarrow \sqrt{112} \approx \frac{1185.292}{112} = 10.582962857\ldots = 10.58296 \text{ (5 d.p.)}$$

d Using a calculator $\sqrt{112} = 10.5830052$ (7 d.p.)

$$\text{Percentage error } = \frac{10.5830052 - 10.5829643}{10.5830052} \times 100 = 0.00039\% \text{ (5 d.p.)}$$

Note, you will get different answers if you use values rounded to 5 d.p. in calculating the percentage error.

26 Expand $(3+2x)^{-3}$ using the binomial expansion:

$$(3+2x)^{-3} = 3^{-3}\left(1+\frac{2}{3}x\right)^{-3}$$

$$= \frac{1}{27}\left(1 + (-3)\left(\frac{2}{3}x\right) + \frac{(-3)(-4)}{2!}\left(\frac{2}{3}x\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{2}{3}x\right)^3 + \ldots\right)$$

$$= \frac{1}{27}\left(1 - 2x + \frac{8}{3}x^2 - \frac{80}{27}x^3 + \ldots\right)$$

So $$(1+x)(3+2x)^{-3} = \frac{1}{27}(1+x)\left(1 - 2x + \frac{8}{3}x^2 - \frac{80}{27}x^3 + \ldots\right)$$

$$= \frac{1}{27}\left(1 + (-2+1)x + \left(\frac{8}{3} - 2\right)x^2 + \left(-\frac{80}{27} + \frac{8}{3}\right)x^3 + \ldots\right)$$

$$= \frac{1}{27} - \frac{1}{27}x + \frac{2}{81}x^2 - \frac{8}{729}x^3 + \ldots$$

27 a $\mathrm{h}(x) = (4-9x)^{\frac{1}{2}} = 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

So using the binomial expansion

$$\mathrm{h}(x) = 2\left(1+\left(\tfrac{1}{2}\right)\left(-\tfrac{9}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\tfrac{9}{4}x\right)^2+\ldots\right)$$

$$= 2\left(1-\tfrac{9}{8}x-\tfrac{81}{128}x^2+\ldots\right)$$

$$= 2-\tfrac{9}{4}x-\tfrac{81}{64}x^2+\ldots$$

b $\mathrm{h}\left(\frac{1}{100}\right) = \left(4-\frac{9}{100}\right)^{\frac{1}{2}} = \left(\frac{400-9}{100}\right)^{\frac{1}{2}} = \frac{\sqrt{391}}{10}$

c $\mathrm{h}\left(\frac{1}{100}\right) \approx 2-\frac{9}{4}\left(\frac{1}{100}\right)-\frac{81}{64}\left(\frac{1}{100}\right)^2 = 1.97737$ (5 d.p.)

28 a $(a+bx)^{-2} = \frac{1}{a^2}\left(1+\frac{b}{a}x\right)^{-2}$

$$= \frac{1}{a^2}\left(1+(-2)\left(\frac{b}{a}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{b}{a}x\right)^2+\ldots\right)$$

$$= \frac{1}{a^2}-\frac{2b}{a^3}x+\frac{3b^2}{a^4}x^2+\ldots$$

$$= \frac{1}{4}+\frac{1}{4}x+cx^2\ldots$$

So $\frac{1}{a^2} = \frac{1}{4} \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$

When $a = 2$, comparing the x coefficient gives

$-\frac{2b}{a^3} = \frac{1}{4} \Rightarrow b = -\frac{a^3}{8} = -1$

Comparing the x^2 coefficient gives

$c = \frac{3b^2}{a^4} = \frac{3}{2^4} = \frac{3}{16}$

So one solution is $a = 2, b = -1, c = \frac{3}{16}$

When $a = -2$, comparing the x coefficient gives

$-\frac{2b}{a^3} = \frac{1}{4} \Rightarrow b = -\frac{a^3}{8} = 1$

Comparing the x^2 coefficient gives

$c = \frac{3b^2}{a^4} = \frac{3}{2^4} = \frac{3}{16}$

So second solution is $a = -2, b = 1, c = \frac{3}{16}$

Note that the two solutions yield the same expression

$$(2-x)^{-2} = \left(-1\times(x-2)\right)^{-2} = (-1)^{-2}(x-2)^{-2} = (x-2)^{-2}$$

28 b Coefficient of x^3 in expansion of $(x-2)^{-2}$

$$\frac{1}{4}\frac{(-2)(-3)(-4)}{3!}\left(-\frac{1}{2}\right)^3 = \frac{1}{8}$$

29 a $\dfrac{3+5x}{(1+3x)(1-x)} = \dfrac{A}{1+3x} + \dfrac{B}{1-x}$

$\Rightarrow 3+5x = A(1-x)+B(1+3x)$

Set $x=1$: $8 = 4B \Rightarrow B = 2$

Set $x=-\frac{1}{3}$: $\frac{4}{3} = \frac{4}{3}A \Rightarrow A = 1$

b $\dfrac{3+5x}{(1+3x)(1-x)} = (1+3x)^{-1} + 2(1-x)^{-1}$

$$= \left(1+(-1)(3x)+\frac{(-1)(-2)}{2!}(3x)^2+\ldots\right)+2\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\ldots\right)$$

$$= (1+2)+(-3x+2x)+(9x^2+2x^2)+\ldots$$

$$= 3-x+11x^2+\ldots$$

30 a $\dfrac{3x-1}{(1-2x)^2} = \dfrac{A}{1-2x} + \dfrac{B}{(1-2x)^2}$

$\Rightarrow 3x-1 = A(1-2x)+B$

Set $x=\frac{1}{2}$: gives B= $\frac{1}{2}$

Compare coefficients of x gives $3 = -2A \Rightarrow A = -\frac{3}{2}$

b $\dfrac{3x-1}{(1-2x)^2} = -\dfrac{3}{2}(1-2x)^{-1} + \dfrac{1}{2}(1-2x)^{-2}$

Expand each term using the bionomial expansion

$$-\frac{3}{2}(1-2x)^{-1} = -\frac{3}{2}\left(1+(-1)(-2x)+\frac{(-1)(-2)}{2!}(-2x)^2+\frac{(-1)(-2)(-3)}{3!}(-2x)^3+\ldots\right)$$

$$\frac{1}{2}(1-2x)^{-2} = \frac{1}{2}\left(1+(-2)(-2x)+\frac{(-2)(-3)}{2!}(-2x)^2+\frac{(-2)(-3)(-4)}{3!}(-2x)^3+\ldots\right)$$

Now sum the expansions

$$-\frac{3}{2}(1-2x)^{-1}+\frac{1}{2}(1-2x)^{-2} = \left(-\frac{3}{2}+\frac{1}{2}\right)+\left(-3x+2x\right)+(-6x^2+6x^2)+(-12x^3+16x^3)+\ldots$$

$$= -1-x+4x^3+\ldots$$

31 a $\mathrm{f}(x)=\dfrac{25}{(3+2x)^2(1-x)}=\dfrac{A}{3+2x}+\dfrac{B}{(3+2x)^2}+\dfrac{C}{1-x}$

$\Rightarrow 25=A(3+2x)(1-x)+B(1-x)+C(3+2x)^2$

Set $x=1$: $25=25C\Rightarrow C=1$

Set $x=-\frac{3}{2}$: $25=\frac{5}{2}B\Rightarrow B=10$

Compare the coefficients of x^2

$0=-2A+4C\Rightarrow A=2C=2$

So $A=2$, $B=10$, $C=1$

b From part (a) $\mathrm{f}(x)=2(3+2x)^{-1}+10(3+2x)^{-2}+(1-x)^{-1}$

$$=\tfrac{2}{3}\left(1+\tfrac{2}{3}x\right)^{-1}+\tfrac{10}{9}\left(1+\tfrac{2}{3}x\right)^{-2}+(1-x)^{-1}$$

Now expand each part of the equation using the binomial expansion

$$\mathrm{f}(x)=\tfrac{2}{3}\left(1+(-1)\left(\tfrac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\tfrac{2}{3}x\right)^2+\ldots\right)+\tfrac{10}{9}\left(1+(-2)\left(\tfrac{2}{3}x\right)+\frac{(-2)(-3)}{2!}\left(\tfrac{2}{3}x\right)^2+\ldots\right)$$

$$+\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\ldots\right)$$

$$=\left(\tfrac{2}{3}+\tfrac{10}{9}+1\right)+\left(-\tfrac{4}{9}x-\tfrac{40}{27}x+x\right)+\left(\tfrac{8}{27}x^2+\tfrac{40}{27}x^2+x^2\right)+\ldots$$

$$=\tfrac{25}{9}-\tfrac{25}{27}x+\tfrac{25}{9}x^2+\ldots$$

32 a $\dfrac{40x^2+30x+31}{(x+4)(2x+3)}=A+\dfrac{B}{x+4}+\dfrac{C}{2x+3}$

$\Rightarrow 4x^2+30x+31=A(x+4)(2x+3)+B(2x+3)+C(x+4)$

Set $x=-4$: $64-120+31=-25=-5B\Rightarrow B=5$

Set $x=-\frac{3}{2}$: $9-45+31=-5=\frac{5}{2}C\Rightarrow C=-2$

Compare coefficients of x^2

$4=2A\Rightarrow A=2$

Solution: $A=2$, $B=5$, $C=-2$

b Expand $\mathrm{f}(x)=2+5(x+4)^{-1}-2(2x+3)^{-1}$

Rewrite as $\mathrm{f}(x)=2+\dfrac{5}{4}\left(1+\dfrac{x}{4}\right)^{-1}-\dfrac{2}{3}\left(1+\dfrac{2}{3}x\right)^{-1}$

$$\mathrm{f}(x)=2+\frac{5}{4}\left(1+(-1)\left(\frac{x}{4}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{4}\right)^2+\ldots\right)-\frac{2}{3}\left(1+(-1)\left(\frac{2}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{2}{3}x\right)^2+\ldots\right)$$

$$=\left(2+\frac{5}{4}-\frac{2}{3}\right)+\left(-\frac{5}{16}x+\frac{4}{9}x\right)+\left(\frac{5}{64}x^2-\frac{8}{27}x^2\right)+\ldots$$

$$=\frac{31}{12}+\frac{19}{144}x-\frac{377}{1728}x^2+\ldots$$

Challenge

1 Assumption: there exists a, $b \in \mathbb{Z}$ such that $a^2 - 8b = 2$
First let $a^2 = 2 + 8b = 2(4b + 1)$ which means that a^2 is even.
Since a^2 is even, it follows that a must also be even.
Then let $a = 2c$ where c is an integer.
So $(2c)^2 = 2(4b + 1)$, or $2c^2 - 4b = 1$
Since we now have $2(c^2 - 2b) = 1$, and that also b, $c \in \mathbb{Z}$ it follows that from our statement that 1 must be even.
Since we know 1 isn't even, this contradicts our assumption, and so $a^2 - 8b \neq 2$

2 $\dfrac{2x^4+3}{x^2-1} = \dfrac{2x^4+3}{(x+1)(x-1)} = Ax^2 + Bx + C + \dfrac{D}{x+1} + \dfrac{E}{x-1}$

$$2x^4+3 = Ax^2(x+1)(x-1) + Bx(x+1)(x-1) + C(x+1)(x-1) + D(x-1) + E(x+1)$$
$$= Ax^4 - Ax^2 + Bx^3 - Bx + Cx^2 - C + Dx - D + Ex + E$$
$$= Ax^4 + Bx^3 + (C-A)x^2 + (D-B+E)x + (E-C-D)$$

Comparing coefficients:
For x^4:
$A = 2$
For x^3:
$B = 0$
For x^2:
$B = 0$
$C - A = 0 \Rightarrow C = 2$
For x:
$D - B + E = 0 \Rightarrow D + E = 0$ **(1)**
For constant:
$E - C - D = 3 \Rightarrow E - D = 5$ **(2)**
Adding **(1)** and **(2)** gives:
$D + E + E - D = 0 + 5$
$2E = 5$
$E = \dfrac{5}{2}$

Substituting $E = \dfrac{5}{2}$ into **(1)** gives:

$D + \left(\dfrac{5}{2}\right) = 0$

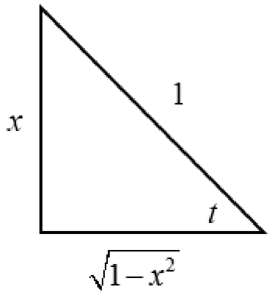
$D = -\dfrac{5}{2}$

Therefore:

$$\frac{2x^4+3}{x^2-1} = 2x^2 + 2 - \frac{5}{2(x+1)} + \frac{5}{2(x-1)}$$

3 a $x = \sin t,\ y = \sin 3t,\ 0 \leqslant t \leqslant \pi$

$$\sin 3t = \sin 2t \cos t + \cos 2t \sin t$$
$$= \cos t\left(2\sin t \cos t\right) + \sin t\left(2\cos^2 t - 1\right)$$
$$= 2\sin t \cos^2 t + 2\sin t \cos^2 t - \sin t$$
$$= 4\sin t \cos^2 t - \sin t \qquad \textbf{(1)}$$

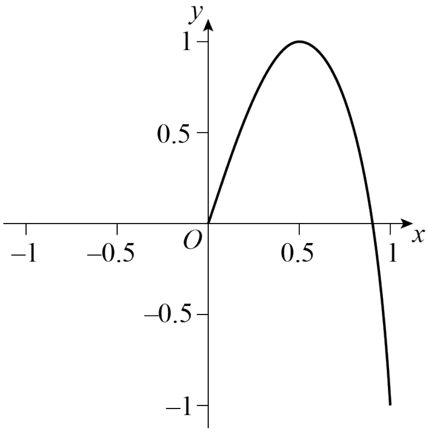


$$\cos t = \frac{\sqrt{1-x^2}}{1} \Rightarrow \cos t = \sqrt{1-x^2}$$

Substituting $y = \sin 3t$, $x = \sin t$ and $\cos t = \sqrt{1-x^2}$ into **(1)** gives:

$$y = 4x\left(\sqrt{1-x^2}\right)^2 - x$$
$$= 4x\left(1-x^2\right) - x$$
$$= 3x - 4x^3$$

b Domain: $0 \leqslant x \leqslant 1$
Range: $-1 \leqslant y \leqslant 1$



4 $(1-2x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)(-2x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-2x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-2x)^3 + \ldots$

$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \ldots$, valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

$(0.98)^{\frac{1}{2}} = (1-0.02)^{\frac{1}{2}}$

Using $(1-2x)^{\frac{1}{2}} = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3$ with $x = 0.01$ gives:

$(0.98)^{\frac{1}{2}} = 1 - 0.01 - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$

$= 0.9899495$

$\sqrt{98} = 98^{\frac{1}{2}} = 100^{\frac{1}{2}} \times 0.98^{\frac{1}{2}}$

$= 10 \times 0.9899495$

$= 9.899495$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sqrt{98} = \sqrt{2 \times 49}$

$= 7\sqrt{2}$

$\sqrt{2} = \frac{\sqrt{98}}{7}$

$\frac{\sqrt{2}}{2} = \frac{\sqrt{98}}{14}$

Therefore:

$\frac{\sqrt{2}}{2} = \frac{9.899495}{14}$

$= 0.707107$ (6 d.p.)