Practice exam paper

1 Assumption: if $n^2 + 1$ is even then n can be even.
Let $n = 2k$ where k is an integer.
Then $(2k)^2 + 1 = 4k^2 + 1$
But $4k^2$ is even, and so $4k^2 + 1$ must be odd.
This contradicts our assumption, and so if $n^2 + 1$ is even then n must be odd.

2 **a** $x^2 + 4xy + y^2 + 1 = 0$ **(1)**

$$2x + 4y + 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2(2x + y)\frac{dy}{dx} = -2(x + 2y)$$

$$\frac{dy}{dx} = -\frac{x + 2y}{2x + y}$$

b When the gradient is parallel to the x-axis $\frac{dy}{dx} = 0$

$$-\frac{x + 2y}{2x + y} = 0 \Rightarrow y = -\frac{1}{2}x$$

Substituting into **(1)** gives:

$$x^2 + 4x\left(-\frac{1}{2}x\right) + \left(-\frac{1}{2}x\right)^2 + 1 = 0$$

$$x^2 - 2x^2 + \frac{1}{4}x^2 + 1 = 0$$

$$-\frac{3}{4}x^2 + 1 = 0$$

$$-3x^2 + 4 = 0$$

$$x^2 = \frac{4}{3}$$

$$x = \pm\frac{2}{\sqrt{3}}$$

When $x = \frac{2}{\sqrt{3}}$, $y = -\frac{1}{\sqrt{3}}$

When $x = -\frac{2}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$

3 a $y = xe^{2x}$

$\frac{dy}{dx} = e^{2x} + 2xe^{2x}$

At the turning point $\frac{dy}{dx} = 0$

$e^{2x} + 2xe^{2x} = 0$

$e^{2x}(1+2x) = 0$

$x = -\frac{1}{2}$

When $x = -\frac{1}{2}$

$y = -\frac{1}{2}e^{2\left(-\frac{1}{2}\right)}$

$= -\frac{1}{2e}$

Therefore the turning point has coordinates $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$

b $V = \pi\int_a^b y^2 \, dx$ where $y = xe^{2x}$

$= \pi\int_1^2 \left(xe^{2x}\right)^2 \, dx$

$= \pi\int_1^2 x^2e^{4x} \, dx$

$\int_1^2 x^2e^{4x} \, dx$ is of the form $\int u\frac{dv}{dx} \, dx = uv - \int v\frac{du}{dx} \, dx$

with $u = x^2 \Rightarrow \frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^{4x} \Rightarrow v = \frac{1}{4}e^{4x}$

$\int x^2e^{4x} \, dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\int xe^{4x} \, dx$

$\int xe^{4x} \, dx$ is of the form $\int u\frac{dv}{dx} \, dx = uv - \int v\frac{du}{dx} \, dx$

with $u = x \Rightarrow \frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^{4x} \Rightarrow v = \frac{1}{4}e^{4x}$

$\int x^2e^{4x} \, dx = \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left(\frac{1}{4}xe^{4x} - \frac{1}{4}\int e^{4x} \, dx\right)$

$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8}\int e^{4x} \, dx$

$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x}(+c)$

Therefore:

$$\pi\int_1^2 x^2e^{4x}\,dx = \pi\left[\frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{32}e^{4x}\right]_1^2$$

$$= \pi\left[\left(e^8 - \frac{1}{4}xe^8 + \frac{1}{32}e^8\right) - \left(e^4 - \frac{1}{4}e^4 + \frac{1}{32}e^4\right)\right]$$

$$= \pi\left(\frac{25}{32}e^8 - \frac{5}{32}e^4\right)$$

$$= \frac{5}{32}e^4\pi\left(5e^4 - 1\right)$$

4 a $y = \cos 2t$

$= \cos^2 t - \sin^2 t$

$= \cos^2 t - \left(1 - \cos^2 t\right)$

$= 2\cos^2 t - 1$

Since $x = \cos t$

$y = 2x^2 - 1$

b Domain: $0 \le t \le 2\pi$

Range: $-1 \le y \le 1$

c $y = 2x^2 - 1$

$\frac{dy}{dx} = 4x$

$= 4\cos t$

Alternative method:

$\frac{dy}{dt} = -2\sin 2t = -4\sin t\cos t$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{-4\sin t\cos t}{-\sin t} = 4\cos t$

d When $t = 0$, gradient of tangent = 4, $x = 1$ and $y = 1$

Therefore the equation of the tangent is:

$y - 1 = 4\left(x - 1\right)$

$y = 4x - 3$

5 a $f\left(x\right) = \frac{1}{\left(1 - 3x\right)^2}$

$f\left(x\right) = \left(1 - 3x\right)^{-2}$

$$\left(1 - 3x\right)^{-2} = 1 + \left(-2\right)\left(-3x\right) + \frac{\left(-2\right)\left(-3\right)}{2!}\left(-3x\right)^2 + \frac{\left(-2\right)\left(-3\right)\left(-4\right)}{3!}\left(-3x\right)^3$$

$$= 1 + 6x + 27x^2 + 108x^3$$

5 b $\frac{1}{(1-3x)^2} = 1 + 6x + 27x^2 + 108x^3$

Therefore:

$$\frac{1}{(0.97)^2} = \frac{1}{(1-3(0.01))^2}$$
$$= 1 + 6(0.01) + 27(0.01)^2 + 108(0.01)^3$$
$$= 1.062808$$

6 a $\frac{dH}{dt} = -20(H-5)$

$$\frac{dH}{H-5} = -20\,dt$$
$$\ln|H-5| = -20t + c$$

When $t = 0$, $H = 40$, therefore:

$$c = \ln 35$$
$$\ln|H-5| = -20t + \ln 35$$
$$\ln\left(\frac{|H-5|}{35}\right) = -20t$$
$$\frac{|H-5|}{35} = e^{-20t}$$
$$H = 5 + 35e^{-20t}$$

b $\frac{dH}{dt} = -20(H-5)$

$$= -20\left(5 + 35e^{-20t} - 5\right)$$
$$= -700e^{-20t}$$

When $t = 0.5$

$$\frac{dH}{dt} = -700e^{-10}$$
$$= -0.31779...$$
$$= 0.0318 \text{ (3 s.f.)}$$

c 5

7 a $A(2, 1, 3)$ and $B(5, -2, 1)$

$$\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -3 \\ -2 \end{pmatrix}$$

7 b If (−4, 7, 7) lies on l then:

$$\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}3\\-3\\-2\end{pmatrix}=\begin{pmatrix}-4\\7\\7\end{pmatrix}$$

$2 + 3t = -4 \Rightarrow t = -2$
$1 - 3t = 7 \Rightarrow t = -2$
$3 - 2t = 7 \Rightarrow t = -2$
Therefore (−4, 7, 7) lies on l.

7 c $\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$$= \frac{\begin{pmatrix}3\\-3\\-2\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{3^2+(-3)^2+(-2)^2}\times\sqrt{1^2}}$$

$$= \frac{3}{\sqrt{22}}$$

$\theta = 50.23...$
$= 50.2°$ (3 s.f.)

d $AC = 3AB$

$$3AB = 3\begin{pmatrix}3\\-3\\-2\end{pmatrix}$$

$$= \begin{pmatrix}9\\-9\\-6\end{pmatrix}$$

Since A has coordinates (2, 1, 3) therefore the possible coordinates of C are:

$$\begin{pmatrix}2\\1\\3\end{pmatrix}+\begin{pmatrix}9\\-9\\-6\end{pmatrix}=\begin{pmatrix}11\\-8\\-3\end{pmatrix} \text{ and } \begin{pmatrix}2\\1\\3\end{pmatrix}-\begin{pmatrix}9\\-9\\-6\end{pmatrix}=\begin{pmatrix}-7\\10\\9\end{pmatrix}$$

8 $x = t^2 \Rightarrow \mathrm{d}x = 2t\,\mathrm{d}t$

$y = 1 - t$

$$V = \pi\int_a^b y^2\,\mathrm{d}x$$

$$V = \pi\int_0^1 (1-t)^2 \cdot 2t\,\mathrm{d}t$$

$$= 2\pi\int_0^1 \left(t - 2t^2 + t^3\right)\mathrm{d}t$$

$$= 2\pi\left[\frac{1}{2}t^2 - \frac{2}{3}t^3 + \frac{1}{4}t^4\right]_0^1$$

$$= 2\pi\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{\pi}{6}$$