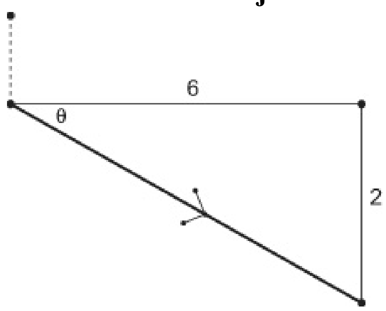
Chapter review 7

1 a $\mathbf{R} = -3\mathbf{i} + 7\mathbf{j} + \mathbf{i} - \mathbf{j}$
$= -2\mathbf{i} + 6\mathbf{j}$
$|\mathbf{R}| = \sqrt{2^2 + 6^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$
The magnitude of **R** is $2\sqrt{10}$ N

b $\tan\theta = \frac{1}{3}$
$\theta = \tan^{-1}\frac{1}{3}$
$= 18°$ (nearest degree)

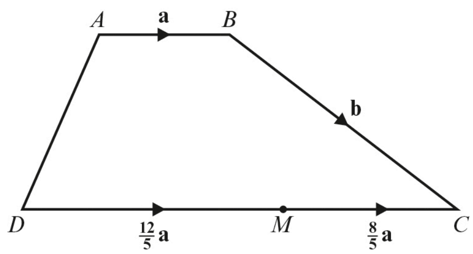
2 a $(\text{Path of } S) = (4\mathbf{i} - 6\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$



$\tan\theta = \frac{1}{3} \Rightarrow \theta = 18.43\ldots°$
$\text{Bearing} = 90° + \theta = 108°$

b Expressing velocity, **v**, in km h^{-1}:
$\mathbf{v} = (6\mathbf{i} - 2\mathbf{j}) \times \frac{60}{40}$
$\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$
Then the speed is:
$\sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10}$
$= 9.49 \text{ km h}^{-1} (3 \text{ s.f.})$

3



a $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$
$= \mathbf{a} + \mathbf{b} - \frac{8}{5}\mathbf{a}$
$= \mathbf{b} - \frac{3}{5}\mathbf{a}$

3 b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{b} - 4\mathbf{a}$

c $\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$
$= \frac{8}{5}\mathbf{a} - \mathbf{b}$

d $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}$
$= 4\mathbf{a} - \mathbf{b} - \mathbf{a}$
$= 3\mathbf{a} - \mathbf{b}$

4 As the vectors are parallel
$5\mathbf{a} + k\mathbf{b} = \frac{5}{8}(8\mathbf{a} + 2\mathbf{b})$
$k = \frac{5}{8} \times 2$
$= \frac{5}{4}$

5 a $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

b $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

c $2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = 2\begin{pmatrix} 7 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} - 3\begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 49 \\ 13 \end{pmatrix}$

6 **a** $4\mathbf{i} - 3\mathbf{j} + 2p\mathbf{i} - p\mathbf{j} = \lambda(2\mathbf{i} - 3\mathbf{j})$

$(4 + 2p)\mathbf{i} - (3 + p)\mathbf{j} = 2\lambda\mathbf{i} - 3\lambda\mathbf{j}$

Equating coefficients:

$4 + 2p = 2\lambda$ and $3 + p = 3\lambda$

Solving simultaneously:

Rearranging the $3 + p = 3\lambda$

$p = 3\lambda - 3$

Using substitution:

$4 + 2(3\lambda - 3) = 2\lambda$

$4 + 6\lambda - 6 = 2\lambda$

$4\lambda = 2$

$\lambda = \frac{1}{2}$

$p = -\frac{3}{2}$

b $\mathbf{a} + \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} - 3\mathbf{i} + \frac{3}{2}\mathbf{j}$

$= \mathbf{i} - \frac{3}{2}\mathbf{j}$

7 $\cos 55° = \dfrac{p}{15}$

$p = 15\cos 55°$

$p = 8.6$

Using Pythagoras' theorem:

$q = \sqrt{15^2 - 8.6^2}$

$= 12.3$

$p = 8.6$ and $q = 12.3$

8 $|3\mathbf{i} - k\mathbf{j}| = \sqrt{3^2 + k^2}$

$= \sqrt{9 + k^2}$

$= 3\sqrt{5}$

$\sqrt{9 + k^2} = \sqrt{45}$

$k^2 + 9 = 45$

$k^2 = 36$

$k = \pm 6$

9 **a** $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$ or $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$\overrightarrow{MN} = \lambda\mathbf{b}$

Using similar triangles:

$\overrightarrow{AN} = \lambda(-\mathbf{a} + \mathbf{b})$

Using $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$

$\overrightarrow{ON} = \mathbf{a} + \lambda(-\mathbf{a} + \mathbf{b}) = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$

Using $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$

$(1 - \lambda)\mathbf{a} + \lambda\mathbf{b} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$

Equating coefficients:

$1 - \lambda = \frac{3}{5}$

$\lambda = \frac{2}{5}$

$\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$

b $\overrightarrow{MN} = \lambda\mathbf{b}$

$= \frac{2}{5}\mathbf{b}$

c $\overrightarrow{AN} = \frac{2}{5}(-\mathbf{a} + \mathbf{b})$

$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$

Therefore, $AN : NB = 2 : 3$

10 Coordinates of M are $(3, 5, 4)$

Distance from M to C

$= \sqrt{(5-3)^2 + (8-5)^2 + (7-4)^2}$

$= \sqrt{4 + 9 + 9} = \sqrt{22}$

11 Distance from P to Q

$= \sqrt{((a-2)-2)^2 + (6-3)^2 + (7-a)^2}$

$= \sqrt{a^2 - 8a + 16 + 9 + 49 - 14a + a^2}$

$= \sqrt{2a^2 - 22a + 74} = \sqrt{14}$

$2a^2 - 22a + 74 = 14$

$a^2 - 11a + 30 = 0$

$(a-5)(a-6) = 0$

$a = 5$ or $a = 6$

12 $|\overrightarrow{AB}| = \sqrt{3^2 + t^2 + 5^2} = \sqrt{t^2 + 34}$

$\sqrt{t^2 + 34} = 5\sqrt{2}$

$t^2 + 34 = 50$

$t^2 = 16$

$t = 4 \quad (\text{since } t > 0)$

So $\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k}$

$= -2\overrightarrow{AB}$

So $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$

13 a Let O be the fixed origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -3\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = -3\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -\mathbf{j} + 5\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{9 + 64 + 9} = \sqrt{82}$

$|\overrightarrow{PR}| = \sqrt{9 + 81 + 64} = \sqrt{154}$

$|\overrightarrow{QR}| = \sqrt{1 + 25} = \sqrt{26}$

$\cos \angle QPR = \dfrac{82 + 154 - 26}{2 \times \sqrt{82} \times \sqrt{154}} = 0.9343...$

$\angle QPR = 20.87...°$

Area of triangle

$= \frac{1}{2} \times \sqrt{82} \times \sqrt{154} \sin 20.87...°$

$= 20.0$ (1 d.p.)

14 a $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = -3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{FD} = \overrightarrow{OD} - \overrightarrow{OF} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}$

b $|\overrightarrow{DE}| = \sqrt{16 + 9 + 16} = \sqrt{41}$

$|\overrightarrow{EF}| = \sqrt{9 + 16 + 16} = \sqrt{41}$

$|\overrightarrow{FD}| = \sqrt{1 + 1 + 64} = \sqrt{66}$

c Two sides are equal in length so the triangle is isosceles.

15 a $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 9\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{81 + 16} = \sqrt{97}$

$|\overrightarrow{PR}| = \sqrt{49 + 1 + 9} = \sqrt{59}$

$|\overrightarrow{QR}| = \sqrt{4 + 25 + 9} = \sqrt{38}$

c $\angle QRP = 90°$ so PQ is the hypotenuse.

$\sin \angle PQR = \dfrac{|\overrightarrow{PR}|}{|\overrightarrow{PQ}|} = \sqrt{\dfrac{59}{97}} = 0.7799...$

$\angle PQR = 51.3°$

16 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -2\mathbf{j} + \mathbf{k}$

$|\overrightarrow{AB}| = \sqrt{1 + 1} = \sqrt{2}$

$|\overrightarrow{BC}| = \sqrt{1 + 9 + 1} = \sqrt{11}$

$|\overrightarrow{AC}| = \sqrt{4 + 1} = \sqrt{5}$

$\cos \angle ABC = \dfrac{2 + 11 - 5}{2 \times \sqrt{2} \times \sqrt{11}} = 0.8528...$

$\angle ABC = 31.5°$

17 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \Rightarrow \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

So $\overrightarrow{BC} = \begin{pmatrix} 9 \\ 10 \\ -6 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{36+4+121} = \sqrt{161}$

$|\overrightarrow{AC}| = \sqrt{225+64+25} = \sqrt{314}$

$|\overrightarrow{BC}| = \sqrt{81+100+36} = \sqrt{217}$

$\cos\angle ABC = \dfrac{161+217-314}{2\times\sqrt{161}\times\sqrt{217}} = 0.1712...$

$\angle ABC = 80.14...°$

Area of triangle ABC
$= \frac{1}{2}\times\sqrt{161}\times\sqrt{217}\times\sin\angle ABC$

Area of parallelogram $ABCD$
$= \sqrt{161}\times\sqrt{217}\times\sin\angle ABC$
$= \sqrt{161}\times\sqrt{217}\times\sin 80.14...°$
$= 184$ (3 s.f.)

18 a $|\overrightarrow{AB}| = \sqrt{4+25+9} = \sqrt{38}$

$|\overrightarrow{AC}| = \sqrt{4+25+9} = \sqrt{38}$

So ABC is an isosceles triangle.
Therefore DBC is an isosceles triangle.

So $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BD}$.

Let O be the fixed origin.

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$
$= \overrightarrow{OC} + \overrightarrow{AB}$
$= \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$
$= \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$

Coordinates of D are $(2, -7, -2)$

b $ABCD$ is a parallelogram with four sides of equal length. It is a rhombus.

c $|\overrightarrow{BC}| = \sqrt{16+36} = \sqrt{52}$

$\cos\angle BAC = \dfrac{38+38-52}{2\times\sqrt{38}\times\sqrt{38}} = 0.3157...$

$\angle BAC = 71.59...°$

Area of triangle ABC
$= \frac{1}{2}\times\sqrt{38}\times\sqrt{38}\times\sin\angle ABC$

Area of parallelogram $ABCD$
$= \sqrt{38}\times\sqrt{38}\times\sin\angle ABC$
$= \sqrt{38}\times\sqrt{38}\times\sin 71.59...°$
$= 36.1$ (3 s.f.)

19 $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\mathbf{c}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{OS} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b})$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{OT} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\mathbf{b}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{OU} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{TU} = \overrightarrow{OU} - \overrightarrow{OT} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Suppose there is a point of intersection, X, of PQ, RS and TU.

$\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

for scalars r, s and t

But $\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$s(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = (r-1)\mathbf{a} + r\mathbf{b} + (1-r)\mathbf{c}$

Comparing coefficients of b and c:
$s = r$ and $s = 1 - r$

Hence $r = s = \frac{1}{2}$

Also $\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{1}{2}t(\mathbf{a} - \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$t(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Hence $t = \frac{1}{2}$

So the point X is the midpoint of all three line segments PQ, RS and TU. Therefore the line segments meet at a point and bisect each other.

20 Total force on particle

$= \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$= ((b+1)\mathbf{i} + (4-b)\mathbf{j} + (7-b)\mathbf{k})$ N

$|\mathbf{F}| = \sqrt{(b+1)^2 + (4-b)^2 + (7-b)^2}$

$= \sqrt{b^2 + 2b + 1 + 16 - 8b + b^2 + 49 - 14b + b^2}$

$= \sqrt{3b^2 - 20b + 66}$

$|\mathbf{F}| = m|\mathbf{a}|$

$\Rightarrow \sqrt{3b^2 - 20b + 66} = 2 \times 3.5 = 7$

$3b^2 - 20b + 66 = 49$

$3b^2 - 20b + 17 = 0$

$(b-1)(3b-17) = 0$

$b = 1$ or $b = \frac{17}{3}$

21 a Air resistance acts in opposition to the motion of the BASE jumper. The motion downwards will be greater than the motion in the other directions.

b Gravitational force downwards
$= 50 \times 9.8 = 490$ N

Total force on BASE jumper
$= \mathbf{W} + \mathbf{F} - 490\mathbf{k}$

$= (16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k})$ N

c $|16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k}| = \sqrt{256 + 169 + 1600}$

$= \sqrt{2025} = 45$ N

Acceleration $= \frac{45}{50} = \frac{9}{10}$ m s^{-2}

Using $s = ut + \frac{1}{2}at^2$:

$180 = 0 + \frac{1}{2} \times \frac{9}{10}t^2$

$t^2 = 400$

$t = 20$

The descent took 20 seconds.

22 a l passes through $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$$\begin{pmatrix}1\\2\\2\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}0\\3\\-1\end{pmatrix}$$

Therefore an equation for l is:

$$\mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}$$

or

$\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{j} - \mathbf{k})$

b When $AC = 2CB$, $\lambda = \frac{2}{3}$

$$\begin{pmatrix}1\\-1\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\3\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\7/3\end{pmatrix}$$

So

$\mathbf{i} + \mathbf{j} + \frac{7}{3}\mathbf{k}$

Or B is the midpoint of AC

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{AB}$$
$$= \begin{pmatrix}1\\2\\2\end{pmatrix} + \begin{pmatrix}0\\3\\-1\end{pmatrix}$$
$$= \begin{pmatrix}1\\5\\1\end{pmatrix}$$

23 $\mathbf{r} = \begin{pmatrix}2\\3\\-4\end{pmatrix} + \lambda\begin{pmatrix}0\\2\\3\end{pmatrix}$

or

$\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + \lambda(2\mathbf{j} + 3\mathbf{k})$

24 When $\lambda = 2$

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + 2(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

$\mathbf{r} = 7\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$

and

$9\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} = 3(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

So parallel

25 a L has position vector $\begin{pmatrix}4\\7\\7\end{pmatrix}$, M has position vector $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and N has position vector $\begin{pmatrix}2\\4\\6\end{pmatrix}$

$$\overrightarrow{ML} = \overrightarrow{OL} - \overrightarrow{OM}$$
$$= \begin{pmatrix}4\\7\\7\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix}$$
$$= \begin{pmatrix}3\\4\\5\end{pmatrix}$$

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$
$$= \begin{pmatrix}2\\4\\6\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix}$$
$$= \begin{pmatrix}1\\1\\4\end{pmatrix}$$

b Let $\angle LMN$ be θ

$$\cos\theta = \frac{\overrightarrow{ML} \cdot \overrightarrow{MN}}{\left|\overrightarrow{ML}\right|\left|\overrightarrow{MN}\right|}$$

$$\overrightarrow{ML} \cdot \overrightarrow{MN} = \begin{pmatrix}3\\4\\5\end{pmatrix} \cdot \begin{pmatrix}1\\1\\4\end{pmatrix}$$
$$= 3(1) + 4(1) + 5(4)$$
$$= 27$$

$$\left|\overrightarrow{ML}\right| = \sqrt{3^2 + 4^2 + 5^2}$$
$$= \sqrt{50}$$

$$\left|\overrightarrow{MN}\right| = \sqrt{1^2 + 1^2 + 4^2}$$
$$= \sqrt{18}$$

Therefore:

$$\cos\theta = \frac{27}{\sqrt{50}\sqrt{18}}$$
$$= \frac{27}{\sqrt{900}}$$
$$= \frac{27}{30} \text{ as required}$$

26 a A has position vector $\begin{pmatrix}9\\-2\\1\end{pmatrix}$,

B has position vector $\begin{pmatrix}6\\2\\6\end{pmatrix}$

and C has position vector $\begin{pmatrix}3\\p\\q\end{pmatrix}$

l passes through A and B, therefore:

$\begin{pmatrix}9\\-2\\1\end{pmatrix}-\begin{pmatrix}6\\2\\6\end{pmatrix}=\begin{pmatrix}3\\-4\\-5\end{pmatrix}$

So a vector equation for l is:

$\mathbf{r}=\begin{pmatrix}9\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix}$

b C lies on l, therefore:

$\begin{pmatrix}9\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix}=\begin{pmatrix}3\\p\\q\end{pmatrix}$

$9+3\lambda=3$

$\Rightarrow \lambda=-2$

$-2-4\lambda=p$

Substituting $\lambda=-2$ gives:

$p=-2-4(-2)$

$=6$

$1-5\lambda=q$

Substituting $\lambda=-2$ gives:

$q=1-5(-2)$

$=11$

So $p=6$ and $q=11$

26 c $\overrightarrow{OC}=\begin{pmatrix}3\\6\\11\end{pmatrix}$ and $\overrightarrow{AB}=\begin{pmatrix}-3\\4\\5\end{pmatrix}$

$\cos\theta=\frac{\overrightarrow{OC}\cdot\overrightarrow{AB}}{|\overrightarrow{OC}||\overrightarrow{AB}|}$

$=\begin{pmatrix}3\\6\\11\end{pmatrix}\cdot\begin{pmatrix}-3\\4\\5\end{pmatrix}$

$=3(-3)+6(4)+11(5)$

$=70$

$\overrightarrow{OC}\cdot\overrightarrow{AB}$

$|\overrightarrow{OC}|=\sqrt{3^2+6^2+11^2}$

$=\sqrt{166}$

$|\overrightarrow{AB}|=\sqrt{(-3)^2+4^2+5^2}$

$=\sqrt{50}$

$\cos\theta=\frac{70}{\sqrt{166}\sqrt{50}}$

$\theta=39.794...$

$=39.8°$ (1 d.p.)

d Let D be the point (x, y, z)

$\overrightarrow{AB}$ has equation $\mathbf{r}=\begin{pmatrix}9\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix}$

$\overrightarrow{OD}$ is perpendicular to $\overrightarrow{AB}$, therefore:

$\begin{pmatrix}3\\-4\\-5\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$

$3x-4y-5z=0$ **(1)**

Since D lies on AB:

$\begin{pmatrix}9\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-4\\-5\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\begin{pmatrix}9+3\lambda\\-2-4\lambda\\1-5\lambda\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$x=9+3\lambda$

$y=-2-4\lambda$

$z=1-5\lambda$

Substituting into **(1)** gives:

$$3(9+3\lambda)-4(-2-4\lambda)-5(1-5\lambda)=0$$
$$27+9\lambda+8+16\lambda-5+25\lambda=0$$
$$50\lambda=-30$$
$$\lambda=\frac{3}{5}$$

Therefore:

$$x=9+3\left(\frac{3}{5}\right)=\frac{36}{5}$$
$$y=-2-4\left(\frac{3}{5}\right)=\frac{2}{5}$$
$$z=1-5\left(\frac{3}{5}\right)=4$$

Hence D has coordinates $\left(\frac{36}{5},\frac{2}{5},4\right)$ and position vector $\frac{36}{5}\mathbf{i}+\frac{2}{5}\mathbf{j}+4\mathbf{k}$

27 a A has position vector $\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and B has position vector $\begin{pmatrix}5\\0\\-3\end{pmatrix}$

l_1 passes through A and B

$$\begin{pmatrix}5\\0\\-3\end{pmatrix}-\begin{pmatrix}1\\2\\-3\end{pmatrix}=\begin{pmatrix}4\\-2\\0\end{pmatrix}$$

So an equation for l_1 is:

$$\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\0\end{pmatrix}$$

b l_2 has the equation $\mathbf{r}=\begin{pmatrix}4\\-4\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\2\end{pmatrix}$

A is the point $(1, 2, -3)$

If A lies on l_2 then:

$$\begin{pmatrix}4\\-4\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$$

$4+\mu=1 \Rightarrow \mu=-3$

$-4-2\mu=2 \Rightarrow \mu=-3$

$3+2\mu=-3 \Rightarrow \mu=-3$

Therefore A lies on l_2

27 c $\begin{pmatrix}4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}=4(1)-2(-2)+0(2)=8$

$$\cos\theta=\frac{l_1\cdot l_2}{|l_1||l_2|}$$
$$|l_1|=\sqrt{4^2+(-2)^2}=\sqrt{20}$$
$$|l_2|=\sqrt{1^2+(-2)^2+2^2}=\sqrt{9}=3$$
$$\cos\theta=\frac{8}{3\sqrt{20}}$$
$$\theta=53.395...=53.4^\circ\ \ (1\text{ d.p.})$$

d C has position vector $\begin{pmatrix}0\\4\\-5\end{pmatrix}$

l_1 has equation $\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\0\end{pmatrix}$

Let the closest point be $D(x, y, z)$.

$$\overrightarrow{CD}=\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}0\\4\\-5\end{pmatrix}=\begin{pmatrix}x\\y-4\\z+5\end{pmatrix}$$

$\overrightarrow{CD}$ is perpendicular to l_1, therefore:

$$\begin{pmatrix}4\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}x\\y-4\\z+5\end{pmatrix}=0$$
$$4x-2(y-4)=0$$
$$4x-2y=-8 \qquad \textbf{(1)}$$

D lies on l_1 therefore:

$$\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\-2\\0\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+4\lambda\\2-2\lambda\\-3\end{pmatrix}$$

Substituting $x = 1 + 4\lambda$ and $y = 2 - 2\lambda$ into **(1)** gives:

$$4(1+4\lambda)-2(2-2\lambda)=-8$$
$$4+16\lambda-4+4\lambda=-8$$
$$20\lambda=-8$$
$$\lambda=-\frac{2}{5}$$

When $\lambda=-\frac{2}{5}$

$$x=1+4\left(-\frac{2}{5}\right)=-\frac{3}{5}$$
$$y=2-2\left(-\frac{2}{5}\right)=\frac{14}{5}$$
$$z=-3$$

The distance, d, between $(0, 4, -5)$ and $\left(-\frac{3}{5},\frac{14}{5},-3\right)$ is found using:

$$d=\sqrt{\left(0-\frac{3}{5}\right)^2+\left(4-\frac{14}{5}\right)^2+\left(-5-(-3)\right)^2}$$
$$=\sqrt{\left(-\frac{3}{5}\right)^2+\left(\frac{6}{5}\right)^2+(-2)^2}$$
$$=\sqrt{\frac{29}{5}}$$
$$=\frac{\sqrt{145}}{5}$$

Alternatively, you can use trigonometry as the shortest distance from C to the line l_1 forms a right-angled triangle, so

$$d=\left|\overrightarrow{CA}\right|\times\sin\theta$$
$$\left|\overrightarrow{CA}\right|=\sqrt{(1-0)^2+(2-4)^2+(-3--5)^2}$$
$$=3$$

Using part **c**

$$\cos^2\theta+\sin^2\theta=1$$
$$\left(\frac{8}{3\sqrt{20}}\right)^2+\sin^2\theta=1$$
$$\sin\theta=\sqrt{1-\frac{16}{45}}$$
$$=\frac{\sqrt{145}}{15}$$
$$d=3\times\frac{\sqrt{145}}{15}$$
$$=\frac{\sqrt{145}}{5}$$

28 a l_1 has equation $\mathbf{r}=\begin{pmatrix}3\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$ and l_2 has equation $\mathbf{r}=\begin{pmatrix}9\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-1\end{pmatrix}$

$$\begin{pmatrix}1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-1\end{pmatrix}=1(4)-2(1)+2(-1)$$
$$=0$$

Therefore the submarines are moving perpendicularly to each other.

b At A:

$$\begin{pmatrix}3\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}9\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-1\end{pmatrix}$$

$3+\lambda=9+4\mu\Rightarrow\lambda-4\mu=6$ **(1)**

$4-2\lambda=1+\mu\Rightarrow2\lambda+\mu=3$ **(2)**

$-5+2\lambda=-2-\mu\Rightarrow2\lambda+\mu=3$ **(3)**

Adding 4 × **(2)** and **(1)** gives:

$$8\lambda+4\mu+\lambda-4\mu=12+6$$
$$9\lambda=18$$
$$\lambda=2$$

Substituting $\lambda=2$ into $\begin{pmatrix}3\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}$ gives:

$$\begin{pmatrix}3\\4\\-5\end{pmatrix}+2\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}5\\0\\-1\end{pmatrix}$$

So a position vector for A is $5\mathbf{i}-\mathbf{k}$

28 c B has position vector $10\mathbf{j} - 11\mathbf{k}$

If l_1 passes through B then:

$$\begin{pmatrix}3\\4\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}0\\10\\-11\end{pmatrix}$$

$3+\lambda=0 \Rightarrow \lambda=-3$

$4-2\lambda=10 \Rightarrow \lambda=-3$

$-5+2\lambda=-11 \Rightarrow \lambda=-3$

Therefore l_1 passes through B

If l_2 passes through B then:

$$\begin{pmatrix}9\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}4\\1\\-1\end{pmatrix}=\begin{pmatrix}0\\10\\-11\end{pmatrix}$$

$9+4\mu=0 \Rightarrow \mu=-\frac{9}{4}$

$1+\mu=10 \Rightarrow \mu=9$

Therefore l_2 does not pass through B

d A is the point (5, 0, −1) and B is the point (0, 10, −11)

Let d be the distance between A and B

$$|d|=\sqrt{(5-0)^2+(0-10)^2+(-1-(-11))^2}$$

$$=\sqrt{5^2+(-10)^2+10^2}$$

$$=\sqrt{225}$$

$$=15$$

Since 1 unit = 100 m

AB has length 1500 m=1.5km

29 a l_1 has equation $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ and l_2 has equation $\mathbf{r}=\begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix}$

If the lines intersect then:

$$\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}1\\4\\-4\end{pmatrix}+\mu\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

$1+2\lambda=1-3\mu \Rightarrow 2\lambda+3\mu=0$ **(1)**

$1+\lambda=4 \Rightarrow \lambda=3$

$-2\lambda=-4+\mu \Rightarrow 2\lambda+\mu=4$ **(2)**

Substituting $\lambda=3$ into **(1)** gives:

$2(3)+3\mu=0$

$\mu=-2$

Substituting $\lambda=3$ and $\mu=-2$ into **(2)** gives:

$\text{LHS}=2(3)+(-2)$

$=4$

$=\text{RHS}$

Therefore the lines intersect.

b Substituting into $\lambda=3$ into $\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ gives:

$$\begin{pmatrix}1\\1\\0\end{pmatrix}+3\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}7\\4\\-6\end{pmatrix}$$

Therefore the position vector of the point of intersection is $7\mathbf{i}+4\mathbf{j}-6\mathbf{k}$

29 c $\cos\theta = \frac{l_1 \cdot l_2}{|l_1||l_2|}$

$l_1 \cdot l_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$= 2(-3) + 1(0) - 2(1)$

$= -8$

$|l_1| = \sqrt{2^2 + 1^2 + (-2)^2}$

$= \sqrt{9}$

$= 3$

$|l_2| = \sqrt{(-3)^2 + 1^2}$

$= \sqrt{10}$

$\cos\theta = -\frac{8}{3\sqrt{10}}$

$= -\frac{4\sqrt{10}}{15}$

Therefore for the acute angle between l_1 and l_2:

$\cos\theta = \frac{4\sqrt{10}}{15}$

30 a l_1 has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

A is the point (3, *a*, 2) and *B* is the point (8, 6, *b*) and *A* and *B* lie on l_1

If *A* lies on l_1 then:

$\begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ a \\ 2 \end{pmatrix}$

$6 + \lambda = 3 \Rightarrow \lambda = -3$

$8 - \lambda = a \Rightarrow 8 - (-3) = a \Rightarrow a = 11$

If *B* lies on l_1 then:

$\begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ b \end{pmatrix}$

$6 + \lambda = 8 \Rightarrow \lambda = 2$

$5 + \lambda = b \Rightarrow b = 5 + 2 \Rightarrow b = 7$

So $a = 11$ and $b = 7$

30 b Let *P* be the point (*x*, *y*, *z*)

l_1 has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\overrightarrow{OP}$ is perpendicular to l_1, therefore:

$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$x - y + z = 0$ **(1)**

Since *P* lies on l_1:

$\begin{pmatrix} 6 \\ 8 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$x = 6 + \lambda$

$y = 8 - \lambda$

$z = 5 + \lambda$

Substituting for *x*, *y* and *z* in **(1)** gives:

$(6 + \lambda) - (8 - \lambda) + (5 + \lambda) = 0$

$3\lambda = -3$

$\lambda = -1$

When $\lambda = -1$

$x = 6 + (-1) = 5$

$y = 8 - (-1) = 9$

$z = 5 + (-1) = 4$

So *P* has coordinates (5, 9, 4)

c $|\overrightarrow{OP}| = \sqrt{5^2 + 9^2 + 4^2}$

$= \sqrt{122}$

31 a *A* has position vector $\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$ and *B* has position vector $\begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

31 b $\mathbf{r} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

c C has position vector $\begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix}$

Let P be the point (x, y, z)

$\overrightarrow{CP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 4 \\ 10 \\ 2 \end{pmatrix} = \begin{pmatrix} x-4 \\ y-10 \\ z-2 \end{pmatrix}$

$\overrightarrow{CP}$ is perpendicular to l, therefore:

$\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} x-4 \\ y-10 \\ z-2 \end{pmatrix} = 0$

$-(x-4)-(y-10)+2(z-2)=0$

$x+y-2z=10$ **(1)**

Since P lies on l_1:

$\begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$x = 6-\lambda$

$y = 3-\lambda$

$z = 4+2\lambda$

Substituting for x, y and z in **(1)** gives:

$(6-\lambda)+(3-\lambda)-2(4+2\lambda)=10$

$6\lambda = -9$

$\lambda = -\frac{3}{2}$

When $\lambda = -\frac{3}{2}$

$x = 6-\left(-\frac{3}{2}\right) = \frac{15}{2}$

$y = 3-\left(-\frac{3}{2}\right) = \frac{9}{2}$

$z = 4+2\left(-\frac{3}{2}\right) = 1$

So P has coordinates (7.5, 4.5, 1)

32 a l_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and l_2 has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 12 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

If l_1 and l_2 meet then:

$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$3+2\lambda = 1+\mu \Rightarrow 2\lambda - \mu = -2$ **(1)**

$-2+\lambda = 12-2\mu \Rightarrow \lambda + 2\mu = 14$ **(2)**

$4-\lambda = 8-\mu \Rightarrow -\lambda + \mu = 4$ **(3)**

Adding **(2)** and **(3)** gives:

$\lambda + 2\mu - \lambda + \mu = 14+4$

$3\mu = 18$

$\mu = 6$

Substituting $\mu = 6$ into **(3)** gives:

$-\lambda + (6) = 4$

$\lambda = 2$

Check by substituting $\mu = 6$ and $\lambda = 2$ into **(1)**:

$\text{LHS} = 2(2)-(6)$

$= -2$

$= \text{RHS}$

Therefore l_1 and l_2 intersect.

Substituting $\lambda = 2$ into $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ gives:

$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$

Therefore A has coordinates (7, 0, 2)

32 b $\cos\theta = \frac{l_1 \cdot l_2}{|l_1||l_2|}$

$l_1 \cdot l_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$= 2(1) + 1(-2) - 1(-1)$

$= 1$

$|l_1| = \sqrt{2^2 + 1^2 + (-1)^2}$

$= \sqrt{6}$

$|l_2| = \sqrt{1^2 + (-2)^2 + (-1)^2}$

$= \sqrt{6}$

$\cos\theta = \frac{1}{\sqrt{6}\sqrt{6}}$

$= \frac{1}{6}$

$\theta = 80.405...$

$= 80.4°$ (1 d.p.)

c B has position vector $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

l_1 has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

If B lies on l_1 then:

$\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

$3 + 2\lambda = 5 \Rightarrow \lambda = 1$

$-2 + \lambda = -1 \Rightarrow \lambda = 1$

$4 - \lambda = 3 \Rightarrow \lambda = 1$

Therefore B lies on l_1.

32 d B has position vector $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$

Using trigonometry, shortest distance from B line l_2 is

$d = |\overrightarrow{BA}| \times \sin\theta$

$|\overrightarrow{BA}| = \sqrt{(7-5)^2 + (0--1)^2 + (2-3)^2}$

$= \sqrt{6}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta + \left(\frac{1}{6}\right)^2 = 1$

$\sin\theta = \sqrt{1 - \left(\frac{1}{6}\right)^2}$

$= \frac{\sqrt{35}}{6}$

So shortest distance is

$d = \sqrt{6} \times \frac{\sqrt{35}}{6}$

$= 2.42$ (3 s.f.)

Alternatively:

Let the closest point on l_2 be $P(x, y, z)$

$\overrightarrow{BP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} x-5 \\ y+1 \\ z-3 \end{pmatrix}$

$\overrightarrow{BP}$ is perpendicular to l_2, therefore:

$\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-5 \\ y+1 \\ z-3 \end{pmatrix} = 0$

$(x-5) - 2(y+1) - (z-3) = 0$

$x - 2y - z = 4$ **(1)**

Since P lies on l_2:

$\begin{pmatrix} 1 \\ 12 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$x = 1 + \mu$

$y = 12 - 2\mu$

$z = 8 - \mu$

Substituting for x, y and z in **(1)** gives:

$(1+\mu) - 2(12 - 2\mu) - (8 - \mu) = 4$

$-31 + 6\mu = 4$

$\mu = \frac{35}{6}$

When $\mu = \frac{35}{6}$

$x = 1 + \left(\frac{35}{6}\right) = \frac{41}{6}$

$y = 12 - 2\left(\frac{35}{6}\right) = \frac{1}{3}$

$z = 8 - \left(\frac{35}{6}\right) = \frac{13}{6}$

So P has coordinates $\left(\frac{41}{6}, \frac{1}{3}, \frac{13}{6}\right)$

B has position coordinates $(5, -1, 3)$

$$|\overrightarrow{BP}| = \sqrt{\left(\frac{41}{6} - 5\right)^2 + \left(\frac{1}{3} - (-1)\right)^2 + \left(\frac{13}{6} - 3\right)^2}$$
$$= \sqrt{\left(\frac{11}{6}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(-\frac{5}{6}\right)^2}$$
$$= \sqrt{\frac{35}{6}}$$
$$= 2.415...$$
$$= 2.42 \text{ (3 s.f.)}$$

33 a A starts at $\begin{pmatrix}120\\-80\\13\end{pmatrix}$ and travels to $\begin{pmatrix}200\\20\\5\end{pmatrix}$

B starts at $\begin{pmatrix}-20\\35\\5\end{pmatrix}$ and travels in the direction $\begin{pmatrix}10\\-2\\0.1\end{pmatrix}$

For A:

$$\begin{pmatrix}120\\-80\\13\end{pmatrix} - \begin{pmatrix}200\\20\\5\end{pmatrix} = \begin{pmatrix}-80\\-100\\8\end{pmatrix}$$

Therefore an equation for A is:

$$\mathbf{r} = \begin{pmatrix}120\\-80\\13\end{pmatrix} + \lambda\begin{pmatrix}-80\\-100\\8\end{pmatrix}$$

An equation for B is:

$$\mathbf{r} = \begin{pmatrix}-20\\35\\5\end{pmatrix} + \mu\begin{pmatrix}10\\-2\\0.1\end{pmatrix}$$

If the paths of the aeroplanes intersect then:

$$\begin{pmatrix}120\\-80\\13\end{pmatrix} + \lambda\begin{pmatrix}-80\\-100\\8\end{pmatrix} = \begin{pmatrix}-20\\35\\5\end{pmatrix} + \mu\begin{pmatrix}10\\-2\\0.1\end{pmatrix}$$

$120 - 80\lambda = -20 + 10\mu$

$\Rightarrow 80\lambda + 10\mu = 140$ **(1)**

$-80 - 100\lambda = 35 - 2\mu$

$\Rightarrow 100\lambda - 2\mu = -115$ **(2)**

$13 + 8\lambda = 5 + 0.1\mu$

$\Rightarrow 8\lambda - 0.1\mu = -8$ **(3)**

Adding 5 × **(2)** and **(1)** gives:

$500\lambda - 10\mu + 80\lambda + 10\mu = -575 + 140$

$580\lambda = -435$

$\lambda = -\frac{3}{4}$

Substituting $\lambda = -\frac{3}{4}$ into **(1)** gives:

$80\left(-\frac{3}{4}\right) + 10\mu = 140$

$10\mu = 200$

$\mu = 20$

Substituting $\lambda = -\frac{3}{4}$ and $\mu = 20$ into **(1)** gives:

$$\text{LHS} = 80\left(-\frac{3}{4}\right) + 10(20)$$
$$= -60 + 200$$
$$= 140$$
$$= \text{RHS}$$

Therefore the paths of the aeroplanes intersect.

Substituting $\mu = 20$ into $\begin{pmatrix}-20\\35\\5\end{pmatrix} + \mu\begin{pmatrix}10\\-2\\0.1\end{pmatrix}$ gives:

$$\begin{pmatrix}-20\\35\\5\end{pmatrix} + 20\begin{pmatrix}10\\-2\\0.1\end{pmatrix} = \begin{pmatrix}180\\-5\\7\end{pmatrix}$$

So the paths intersect at the point $(180, -5, 7)$.

b The aeroplanes don't necessarily pass through $(180, -5, 7)$ at the same time.