Exercise 7K

1 $|\mathbf{a}| = 3$, $|\mathbf{b}| = 3$ and $\theta = 60°$

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$
$$= 3\times3\times\cos 60$$
$$= 4.5$$

2 a $\mathbf{a} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\begin{pmatrix}5\\2\\3\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-1\end{pmatrix} = 5(2)+2(-1)+3(-1)$$
$$= 2$$

b $\mathbf{a} = 10\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j} - 12\mathbf{k}$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\begin{pmatrix}10\\-7\\4\end{pmatrix}\cdot\begin{pmatrix}3\\-5\\-12\end{pmatrix} = 10(3)-7(-5)+4(-12)$$
$$= 17$$

c $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\begin{pmatrix}1\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\4\end{pmatrix} = 1(-1)+1(-1)-1(4)$$
$$= -6$$

d $\mathbf{a} = 2\mathbf{i} - \mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 5\mathbf{j} - 8\mathbf{k}$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}6\\-5\\-8\end{pmatrix} = 2(6)+0(-5)-1(-8)$$
$$= 20$$

e $\mathbf{a} = 3\mathbf{j} + 9\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 12\mathbf{j} + 4\mathbf{k}$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
$$\begin{pmatrix}0\\3\\9\end{pmatrix}\cdot\begin{pmatrix}1\\-12\\4\end{pmatrix} = 0(1)+3(-12)+9(4)$$
$$= 0$$

3 a $\mathbf{a} = 3\mathbf{i} + 7\mathbf{j}$, $\mathbf{b} = 5\mathbf{i} + \mathbf{j}$

$$|\mathbf{a}| = \sqrt{3^2+7^2}$$
$$= \sqrt{58}$$
$$|\mathbf{b}| = \sqrt{5^2+1^2}$$
$$= \sqrt{26}$$
$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$$
$$\begin{pmatrix}3\\7\end{pmatrix}\cdot\begin{pmatrix}5\\1\end{pmatrix} = 3(5)+7(1)$$
$$= 22$$
$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{22}{\sqrt{58}\sqrt{26}}$$
$$\theta = 55.491\ldots$$
$$= 55.5° \text{ (1 d.p.)}$$

b $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$, $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$

$$|\mathbf{a}| = \sqrt{2^2+(-5)^2}$$
$$= \sqrt{29}$$
$$|\mathbf{b}| = \sqrt{6^2+3^2}$$
$$= \sqrt{45}$$
$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$$
$$\begin{pmatrix}2\\-5\end{pmatrix}\cdot\begin{pmatrix}6\\3\end{pmatrix} = 2(6)-5(3)$$
$$= -3$$
$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= -\frac{3}{\sqrt{29}\sqrt{45}}$$
$$\theta = 94.763\ldots$$
$$= 94.8° \text{ (1 d.p.)}$$

3 c $\mathbf{a} = \mathbf{i} - 7\mathbf{j} + 8\mathbf{k}$, $\mathbf{b} = 12\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{1^2 + (-7)^2 + 8^2} = \sqrt{114}$$

$$|\mathbf{b}| = \sqrt{12^2 + 2^2 + 1^2} = \sqrt{149}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}1\\-7\\8\end{pmatrix}\cdot\begin{pmatrix}12\\2\\1\end{pmatrix} = 1(12) - 7(2) + 8(1) = 6$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{6}{\sqrt{114}\sqrt{149}}$$

$$\theta = 87.361\ldots = 87.4^\circ \text{ (1 d.p.)}$$

d $\mathbf{a} = -\mathbf{i} - \mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

$$|\mathbf{a}| = \sqrt{(-1)^2 + (-1)^2 + 5^2} = \sqrt{27}$$

$$|\mathbf{b}| = \sqrt{11^2 + (-3)^2 + 4^2} = \sqrt{146}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}-1\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}11\\-3\\4\end{pmatrix} = -1(11) - 1(-3) + 5(4) = 12$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{12}{\sqrt{27}\sqrt{146}}$$

$$\theta = 78.981\ldots = 79.0^\circ \text{ (1 d.p.)}$$

3 e $\mathbf{a} = 6\mathbf{i} - 7\mathbf{j} + 12\mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$|\mathbf{a}| = \sqrt{6^2 + (-7)^2 + 12^2} = \sqrt{229}$$

$$|\mathbf{b}| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}6\\-7\\12\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix} = 6(-2) - 7(1) + 12(1) = -7$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -\frac{7}{\sqrt{229}\sqrt{6}}$$

$$\theta = 100.885\ldots = 100.9^\circ \text{ (1 d.p.)}$$

f $\mathbf{a} = 4\mathbf{i} + 5\mathbf{k}$, $\mathbf{b} = 6\mathbf{i} - 2\mathbf{j}$

$$|\mathbf{a}| = \sqrt{4^2 + 5^2} = \sqrt{41}$$

$$|\mathbf{b}| = \sqrt{6^2 + (-2)^2} = \sqrt{40}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}4\\0\\5\end{pmatrix}\cdot\begin{pmatrix}6\\-2\\0\end{pmatrix} = 4(6) - 0(-2) + 5(0) = 24$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{24}{\sqrt{41}\sqrt{40}}$$

$$\theta = 53.655\ldots = 53.7^\circ \text{ (1 d.p.)}$$

3 g $\mathbf{a} = -5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}$

$|\mathbf{a}| = \sqrt{(-5)^2 + 2^2 + (-3)^2}$

$= \sqrt{38}$

$|\mathbf{b}| = \sqrt{2^2 + (-2)^2 + (-11)^2}$

$= \sqrt{129}$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -11 \end{pmatrix} = -5(2) + 2(-2) - 3(-11)$$

$= 19$

$\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$= \dfrac{19}{\sqrt{38}\sqrt{129}}$

$\theta = 74.254...$

$= 74.3°$ (1 d.p.)

h $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{3}$

$|\mathbf{b}| = \sqrt{1^2 + (-1)^2 + (1)^2}$

$= \sqrt{3}$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 1(1) + 1(-1) + 1(1)$$

$= 1$

$\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$= \dfrac{1}{\sqrt{3}\sqrt{3}}$

$\theta = 70.528...$

$= 70.5°$ (1 d.p.)

4 a Let $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$ and let $\mathbf{b} = \lambda\mathbf{i} + 6\mathbf{j}$

$\mathbf{a}.\mathbf{b} = 0$

Therefore:

$$\begin{pmatrix} 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ 6 \end{pmatrix} = 0$$

$3(\lambda) + 5(6) = 0$

$\lambda = -10$

b Let $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and let $\mathbf{b} = \lambda\mathbf{i} - 4\mathbf{j} - 14\mathbf{k}$

$\mathbf{a}.\mathbf{b} = 0$

Therefore:

$$\begin{pmatrix} 2 \\ 6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ -4 \\ -14 \end{pmatrix} = 0$$

$2(\lambda) + 6(-4) - 1(-14) = 0$

$\lambda = 5$

c Let $\mathbf{a} = 3\mathbf{i} + \lambda\mathbf{j} - 8\mathbf{k}$ and let $\mathbf{b} = 7\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

$\mathbf{a}.\mathbf{b} = 0$

Therefore:

$$\begin{pmatrix} 3 \\ \lambda \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 1 \end{pmatrix} = 0$$

$3(7) + \lambda(-5) - 8(1) = 0$

$21 - 5\lambda - 8 = 0$

$\lambda = \dfrac{13}{5}$

d Let $\mathbf{a} = 9\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and let $\mathbf{b} = \lambda\mathbf{i} + \lambda\mathbf{j} + 3\mathbf{k}$

$\mathbf{a}.\mathbf{b} = 0$

Therefore:

$$\begin{pmatrix} 9 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \lambda \\ 3 \end{pmatrix} = 0$$

$9(\lambda) - 3(\lambda) + 5(3) = 0$

$6\lambda + 15 = 0$

$\lambda = -\dfrac{5}{2}$

e Let $\mathbf{a} = \lambda\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and let $\mathbf{b} = \lambda\mathbf{i} + \lambda\mathbf{j} + 5\mathbf{k}$

$\mathbf{a}.\mathbf{b} = 0$

Therefore:

$$\begin{pmatrix} \lambda \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} \lambda \\ \lambda \\ 5 \end{pmatrix} = 0$$

$\lambda(\lambda) + 3(\lambda) - 2(5) = 0$

$\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

$\lambda = 2$ or $\lambda = -5$

5 a The positive x-axis is **i**

Let **a** = **i** and let **b** = 9**i** − 5**j** + 3**k**

$$|\mathbf{a}| = \sqrt{1^2} = 1$$

$$|\mathbf{b}| = \sqrt{9^2 + (-5)^2 + 3^2} = \sqrt{115}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}9\\-5\\3\end{pmatrix} = 1(9)+0(-5)+0(3) = 9$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{9}{\sqrt{115}}$$

$$\theta = 32.938... = 32.9° \text{ (1 d.p.)}$$

b The positive y-axis is **j**

Let **a** = **j** and let **b** = 9**i** − 5**j** + 3**k**

$$|\mathbf{a}| = \sqrt{1^2} = 1$$

$$|\mathbf{b}| = \sqrt{9^2 + (-5)^2 + 3^2} = \sqrt{115}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}9\\-5\\3\end{pmatrix} = 0(9)+1(-5)+0(3) = -5$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -\frac{5}{\sqrt{115}}$$

$$\theta = 117.791... = 117.8° \text{ (1 d.p.)}$$

6 a The positive y-axis is **j**

Let **a** = **j** and let **b** = **i** + 11**j** − 4**k**

$$|\mathbf{a}| = \sqrt{1^2} = 1$$

$$|\mathbf{b}| = \sqrt{1^2 + 11^2 + (-4)^2} = \sqrt{138}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\11\\-4\end{pmatrix} = 0(1)+1(11)+0(-4) = 11$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{11}{\sqrt{138}}$$

$$\theta = 20.547... = 20.5° \text{ (1 d.p.)}$$

b The positive z-axis is **k**

Let **a** = **j** and let **b** = **i** + 11**j** − 4**k**

$$|\mathbf{a}| = \sqrt{1^2} = 1$$

$$|\mathbf{b}| = \sqrt{1^2 + 11^2 + (-4)^2} = \sqrt{138}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\11\\-4\end{pmatrix} = 0(9)+0(11)+1(-4) = -4$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -\frac{4}{\sqrt{138}}$$

$$\theta = 109.907... = 109.9° \text{ (1 d.p.)}$$

7 Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and let $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

$|\mathbf{a}| = \sqrt{1^2 + 1^2 + 1^2}$

$= \sqrt{3}$

$|\mathbf{b}| = \sqrt{2^2 + 1^2 + 1^2}$

$= \sqrt{6}$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 1(2) + 1(1) + 1(1)$

$= 4$

$\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$= \dfrac{4}{\sqrt{3}\sqrt{6}}$

$= \dfrac{2\sqrt{2}}{3}$

8 Let $\mathbf{a} = \mathbf{i} + 3\mathbf{j}$ and let $\mathbf{b} = \mathbf{j} + \lambda\mathbf{k}$

The angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°

$|\mathbf{a}| = \sqrt{1^2 + 3^2}$

$= \sqrt{10}$

$|\mathbf{b}| = \sqrt{1^2 + \lambda^2}$

$= \sqrt{\lambda^2 + 1}$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$\begin{pmatrix}1\\3\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\k\end{pmatrix} = 1(0) + 3(1) + 0(k)$

$= 3$

$\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$= \dfrac{3}{\sqrt{10}\sqrt{\lambda^2+1}}$

Since $\theta = 60°$

$\dfrac{3}{\sqrt{10}\sqrt{\lambda^2+1}} = \dfrac{1}{2}$

$\sqrt{10}\sqrt{\lambda^2+1} = 6$

$10(\lambda^2 + 1) = 36$

$\lambda^2 + 1 = \dfrac{18}{5}$

$\lambda^2 = \dfrac{13}{5}$

$\lambda = \pm\sqrt{\dfrac{13}{5}}$ as required

9 a Let **a** = **i** + **j** – 3**k**, let **b** = 5**i** – 2**j** – **k** and let the perpendicular vector be **c** = x**i** + y**j** + z**j**

$$\mathbf{a}\cdot\mathbf{c}=\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$x+y-3z=0$ **(1)**

$$\mathbf{b}\cdot\mathbf{c}=\begin{pmatrix}5\\-2\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$5x-2y-z=0$ **(2)**

Substituting $z = 1$ into **(1)** and **(2)** gives:

$x+y=3$ **(3)**

$5x-2y=1$ **(4)**

Adding 2 × **(3)** to **(4)** gives:

$$2x+2y+5x-2y=6+1$$
$$7x=7$$
$$x=1$$

Substituting $x = 1$ into **(3)** gives:

$(1)+y=3$

$y=2$

So a possible vector is:

i + 2**j** + **k**

9 b Let **a** = 2**i** + 3**j** – 4**k**, let **b** = **i** – 6**j** + 3**k** and let the perpendicular vector be **c** = x**i** + y**j** + z**j**

$$\mathbf{a}\cdot\mathbf{c}=\begin{pmatrix}2\\3\\-4\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$2x+3y-4z=0$ **(1)**

$$\mathbf{b}\cdot\mathbf{c}=\begin{pmatrix}1\\-6\\3\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix}=0$$

$x-6y+3z=0$ **(2)**

Substituting $z = 1$ into **(1)** and **(2)** gives:

$2x+3y=4$ **(3)**

$x-6y=-3$ **(4)**

Adding 2 × **(3)** to **(4)** gives:

$$4x+6y+x-6y=8-3$$
$$5x=5$$
$$x=1$$

Substituting $x = 1$ into **(3)** gives:

$2(1)+3y=4$

$y=\frac{2}{3}$

So a possible vector is:

$\mathbf{i}+\frac{2}{3}\mathbf{j}+\mathbf{k}$ or 3**i** + 2**j** + 3**k**

9 c Let **a** = 4**i** – 4**j** – **k**, let **b** = –2**i** – 9**j** + 6**k** and let the perpendicular vector be **c** = x**i** + y**j** + z**j**

$$\mathbf{a}\cdot\mathbf{c} = \begin{pmatrix}4\\-4\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$4x - 4y - z = 0$ **(1)**

$$\mathbf{b}\cdot\mathbf{c} = \begin{pmatrix}-2\\-9\\6\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = 0$$

$-2x - 9y + 6z = 0$ **(2)**

Substituting $z = 1$ into **(1)** and **(2)** gives:

$4x - 4y = 1$ **(3)**

$-2x - 9y = -6$ **(4)**

Adding 2 × **(4)** to **(3)** gives:

$$-4x - 18y + 4x - 4y = -12 + 1$$
$$-22y = -11$$
$$y = \frac{1}{2}$$

Substituting $y = \frac{1}{2}$ into **(3)** gives:

$$4x - 4\left(\frac{1}{2}\right) = 1$$
$$x = \frac{3}{4}$$

So a possible vector is:

$\frac{3}{4}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}$ or 3**i** + 2**j** + 4**k**

10 A(2**i** + 5**j** + **k**), B(6**i** + **j** – 2**k**) and O(0, 0, 0)

$$\overrightarrow{OA} = \begin{pmatrix}2\\5\\1\end{pmatrix},\ \overrightarrow{OB} = \begin{pmatrix}6\\1\\-2\end{pmatrix} \text{ and}$$

$$\overrightarrow{BA} = \begin{pmatrix}2\\5\\1\end{pmatrix} - \begin{pmatrix}6\\1\\-2\end{pmatrix} = \begin{pmatrix}-4\\4\\3\end{pmatrix}$$

For the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$$\left|\overrightarrow{OA}\right| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}$$

$$\left|\overrightarrow{OB}\right| = \sqrt{6^2 + (1)^2 + (-2)^2} = \sqrt{41}$$

$$\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}2\\5\\1\end{pmatrix}\cdot\begin{pmatrix}6\\1\\-2\end{pmatrix} = 2(6) + 5(1) + 1(-2) = 15$$

$$\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{15}{\sqrt{30}\sqrt{41}}$$
$$\theta = 64.678...$$
$$= 64.7° \text{ (1 d.p.)}$$

For the angle between $\overrightarrow{OA}$ and $\overrightarrow{BA}$

$$\left|\overrightarrow{OA}\right| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{30}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{(-4)^2 + 4^2 + 3^2} = \sqrt{41}$$

$$\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix}2\\5\\1\end{pmatrix}\cdot\begin{pmatrix}-4\\4\\3\end{pmatrix} = 2(-4) + 5(4) + 1(3) = 15$$

$$\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{15}{\sqrt{30}\sqrt{41}}$$
$$\theta = 64.678...$$
$$= 64.7° \text{ (1 d.p.)}$$

Since angles in a triangle sum to 180° the angle between $\overrightarrow{OB}$ and $\overrightarrow{BA}$ is 50.6°

11 a $A(1, 3, 1)$, $B(2, 7, -3)$ and $C(4, -5, 2)$

$$|AB| = \sqrt{(1-2)^2 + (3-7)^2 + (1-(-3))^2}$$
$$= \sqrt{(-1)^2 + (-4)^2 + 4^2}$$
$$= \sqrt{33}$$
$$|BC| = \sqrt{(2-4)^2 + (7-(-5))^2 + ((-3)-2)^2}$$
$$= \sqrt{(-2)^2 + 12^2 + (-5)^2}$$
$$= \sqrt{173}$$

b $$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix}$$

$$\overrightarrow{CB} = \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} - \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 12 \\ -5 \end{pmatrix}$$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{pmatrix} 1 \\ 4 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 12 \\ -5 \end{pmatrix} = 1(-2) + 4(12) - 4(-5)$$
$$= 66$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{66}{\sqrt{33}\sqrt{173}}$$
$$\theta = 29.131...$$
$$= 29.1° \quad (1 \text{ d.p.})$$

12 a $A(7\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$, $B(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $O(0, 0, 0)$

$$\overrightarrow{OA} = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}, \ \overrightarrow{OB} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

For the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$$|\overrightarrow{OA}| = \sqrt{7^2 + 4^2 + 4^2}$$
$$= \sqrt{81}$$
$$= 9$$
$$|\overrightarrow{OB}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$= \sqrt{9}$$
$$= 3$$

$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

$$\begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 7(2) + 4(2) + 4(1) = 26$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$
$$= \frac{26}{3(9)}$$
$$= \frac{26}{27}$$

b Using $\sin^2\theta + \cos^2\theta = 1$ gives:

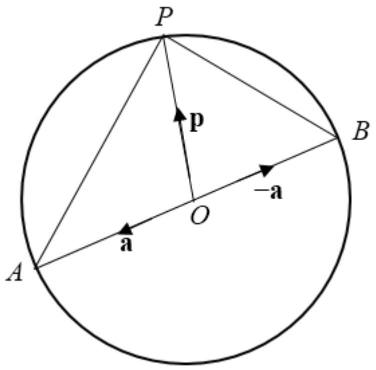
$$\sin^2\theta + \left(\frac{26}{27}\right)^2 = 1$$
$$\sin^2\theta = \frac{53}{729}$$
$$\sin\theta = \pm\frac{\sqrt{53}}{27}$$

Since θ is positive:

$$\sin\theta = \frac{\sqrt{53}}{27}$$

$$\text{Area} = \frac{1}{2} \times |\overrightarrow{OA}| \times |\overrightarrow{OB}| \times \sin\theta$$
$$= \frac{1}{2} \times 9 \times 3 \times \frac{\sqrt{53}}{27}$$
$$= \frac{\sqrt{53}}{2} \quad \text{as required}$$

13 Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OP} = \mathbf{p}$, then $\overrightarrow{OB} = \mathbf{b} = -\mathbf{a}$



$\overrightarrow{AP} = -\mathbf{a} + \mathbf{p}$ and $\overrightarrow{PB} = -\mathbf{p} - \mathbf{a}$

$$\begin{pmatrix} -\mathbf{a} \\ \mathbf{p} \end{pmatrix} \cdot \begin{pmatrix} -\mathbf{p} \\ -\mathbf{a} \end{pmatrix} = -\mathbf{a}(-\mathbf{p}) + \mathbf{p}(-\mathbf{a}) = 0$$

Therefore AP is perpendicular to BP.

14 a $A(5\mathbf{i} - \mathbf{j})$, $B(2\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$ and $C(6\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$$\overrightarrow{CA} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$$

$$\overrightarrow{CB} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix}$$

14 b
$$\left|\overrightarrow{CA}\right| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{17}$$

$$\left|\overrightarrow{CB}\right| = \sqrt{(-4)^2 + 5^2 + 6^2} = \sqrt{77}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 5 \\ 6 \end{pmatrix} = -1(-4) + 0(5) - 4(6) = -20$$

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = -\frac{20}{\sqrt{17}\sqrt{77}} = -\frac{20}{\sqrt{1309}}$$

Using $\sin^2\theta + \cos^2\theta = 1$ gives:

$$\sin^2\theta + \left(-\frac{20}{\sqrt{1309}}\right)^2 = 1$$

$$\sin^2\theta = \frac{909}{1309}$$

$$\sin\theta = \pm\sqrt{\frac{909}{1309}}$$

Since θ is positive:

$$\sin\theta = \sqrt{\frac{909}{1309}}$$

$$\text{Area} = \frac{1}{2} \times \left|\overrightarrow{CA}\right| \times \left|\overrightarrow{CB}\right| \times \sin\theta$$

$$= \frac{1}{2} \times \sqrt{17} \times \sqrt{77} \times \sqrt{\frac{909}{1309}}$$

$$= \frac{1}{2}\sqrt{\frac{1189\,881}{1309}}$$

$$= \frac{\sqrt{909}}{2}$$

$$= \frac{3\sqrt{101}}{2}$$

14 c D can be positioned such that:

$\overrightarrow{BD}$ is parallel to $\overrightarrow{CA}$

$\overrightarrow{CD}$ is parallel to $\overrightarrow{BA}$

$\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$

When $\overrightarrow{BD}$ is parallel to $\overrightarrow{CA}$:

$$\overrightarrow{CA} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{CA}$$

$$= \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix}$$

So D can have coordinates (1, 4, 6)

When $\overrightarrow{CD}$ is parallel to $\overrightarrow{BA}$

$$\overrightarrow{BA} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -10 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{BA}$$

$$= \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ -10 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ -6 \end{pmatrix}$$

So D can have coordinates (9, −6, −6)

When $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ 4 \\ 10 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 10 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{BA}$$

$$= \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 14 \end{pmatrix}$$

So D can have coordinates (3, 4, 14)

d Area of the parallelogram is twice the area of the triangle, so $3\sqrt{101}$

15 a $P(\mathbf{i} - \mathbf{j} + 6\mathbf{k})$, $Q(-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k})$ and $R(3\mathbf{j} - 5\mathbf{k})$

$$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix}$$

$$\overrightarrow{QR} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix}$$

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$\begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -9 \end{pmatrix} = -3(2) + 6(-2) - 2(-9)$$

$$= 0$$

Therefore $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are perpendicular.

b Since $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{QR}$, $\overrightarrow{PR}$ is a diameter.

$$|\overrightarrow{PR}| = \sqrt{(1-0)^2 + (-1-3)^2 + (6-(-5))^2}$$

$$= \sqrt{1^2 + 4^2 + 11^2}$$

$$= \sqrt{138}$$

Therefore the radius is $\dfrac{\sqrt{138}}{2}$

The centre of the circle is the midpoint of $\overrightarrow{PR}$, therefore:

$$\left(\frac{1+0}{2}, \frac{-1+3}{2}, \frac{6-5}{2}\right) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$$

Challenge

1 $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

$\mathbf{b}\cdot\mathbf{a} = |\mathbf{b}||\mathbf{a}|\cos\theta$

So:

$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$

2 a i $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = |\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\theta$

$\cos\theta = \dfrac{PQ}{|\mathbf{b}+\mathbf{c}|}$

Therefore:

$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = |\mathbf{a}|\times PQ$

ii $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\alpha$

$\cos\alpha = \dfrac{PR}{|\mathbf{b}|}$

Therefore:

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\times PR$

iii $\mathbf{a}\cdot\mathbf{c} = |\mathbf{a}||\mathbf{c}|\cos\beta$

$\cos\beta = \dfrac{MN}{|\mathbf{c}|} = \dfrac{RQ}{|\mathbf{c}|}$

Therefore:

$\mathbf{a}\cdot\mathbf{c} = |\mathbf{a}|\times RQ$

b $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = |\mathbf{a}|\times PQ$

$= |\mathbf{a}|\times(PR+RQ)$

$= (|\mathbf{a}|\times PR)+(|\mathbf{a}|\times RQ)$

$= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$